THE CAUCHY PROBLEM FOR THE CH2 SYSTEM

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by

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Abstract

by

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For Sobolev exponent $s > 5/2$, it is shown that the data-to-solution map for the 2-component Camassa-Holm system is continuous from $H^s \times H^{s-1}$ into $C([0, T]; H^s \times H^{s-1})$ but not uniformly continuous. The proof of non-uniform dependence on the initial data is based on the method of approximate solutions, delicate commutator and multiplier estimates, and well-posedness results for the solution and its lifespan. Also, the solution map is Hölder continuous if the $H^s \times H^{s-1}$ norm is replaced by an $H^r \times H^{r-1}$ norm for $0 \leq r < s$. 
DEDICATION

This thesis is dedicated to my family
Donnie, Debra and Katie Thompson.
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CHAPTER 1

HISTORICAL BACKGROUND & DERIVATION OF THE CH2 SYSTEM

In this chapter, we will not only take a journey into the past, but also look at how the 2-component Camassa-Holm system came to be. The reader need only their fond memories of Calculus and a desire to learn a little about the mechanics of fluids. With that being said, strap in and get ready as we begin our quest in search of mathematical enlightenment.

1.1 HISTORICAL BACKGROUND

Look around you. Waves are everywhere. We may only think that waves are limited to the behavior of the oceans, rivers and lakes, but we would be missing so many other venues that waves take their form. The ubiquitous light around us travels on waves. In fact, when light strikes a prism, it disperses into multiple waves of different wave length that give off an arrangement of colors. Waves can also be found in your own DNA. Proteins at the ends of a strain of DNA will often suddenly become energized and create a wave that propagates throughout the double helix. Take a look at nano technology. Information is encoded along a wave and gets trasmitted from
one end of the wire to the other. These are just a few examples of wave phenomena among the many that exist in our world.

We go back, however, to the natural venue for waves: water. Over two thirds of our world is covered in water and about 96 percent is held in our oceans. For thousands of years, the human race has sought to understand the behavior of the oceans and in doing so has led us towards the study of waves. Through the universal language of mathematics, we have made great strides and can now implement models that provide us with a plethora of information on this aforementioned behavior. But how did we get there? Where did these models originate? Who had the mathematical insight to deliver us from the bondage of the qualitative to the quantitative? In the words of Carl Sagan, “join me” and let us embark upon a history of mathematics that will lead us directly from the fundamentals up to today’s most sophisticated models.

As Aretha Franklin once said, “let’s go way on way back when.” But how far back in time do we need to go? The time of Sir Isaac Newton should suffice. This was the man who gave us a better understanding of the cosmos. He was able to mathematically model the motion of the planets and gave the world calculus when he published his Principia in 1687. In this book, he provides us with three fundamental laws of motion. With these laws, mathematicians now had the appropriate tools to create models describing the motions of fluids. As with the introduction of any new material, however, it takes time for the community to digest and disseminate such information. A model based off of Newton’s calculus and his three fundamental laws of motion would not occur until 70 years later.

In 1757, a brilliant Swiss mathematician named Leonard Euler produced a math-
ematical model for the motion of an “ideal fluid.” The fluid was assumed to be nonviscous with no tangential stresses across the boundary of the fluid region. He based this elegant model off of Newton’s second law, “force equals the change in momentum,” conservation of mass and energy. As a result, Euler obtained the following system of equations

\[ \rho (\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u}) = -\nabla p \]
\[ \nabla \cdot \vec{u} = 0, \quad (1.1) \]

where \( \rho \) is the density of the fluid, \( \vec{u} \) is the fluid velocity and \( p \) is the pressure. Problems arise, however, when we look at the type of fluid he considered. Viscosity is a natural physical property that exists in fluids. Furthermore, a fluid that exhibits the absence of tangential forces cannot generate any rotation. If tornados or hurricanes consisted of ideal fluids, then they would be neverending. Thus, these physical properties needed to be included into the mathematical model. Now the only question was: who would do it?

Enter Claude Navier and Sir George Gabriel Stokes. In 1821, Navier included the effects of viscosity into the model and arrived at the following system of equations

\[ \rho (\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u}) = -\nabla p + \nu \Delta \vec{u} \]
\[ \nabla \cdot \vec{u} = 0, \quad (1.2) \]

where \( \nu \) is the coefficient of viscosity. What makes this interesting is that the system of equations he obtained was correct even though the physical assumptions he based
it off of were wrong! In 1845, however, Stokes rederived the above system but in a clear and correct theoretical manner. The Navier-Stokes equations \((1.2)\) are now the basis for modern fluid mechanics and dynamics and much research has been dedicated to this system of nonlinear partial differential equations. In fact, in May of 2000, the Clay Institute of Mathematics announced several “millenium prize problems” one of which is the existence of smooth solutions to the Navier-Stokes equations given smooth initial profile.

From here we will not only show how the Euler equations are derived in the context of an “idealized fluid,” but also create a context for shallow water waves and show how the 2-component Camassa-Holm (CH2) system is obtained directly from the Euler equations. The reader will only need basic principles from calculus and differential equations in order to fully comprehend the derivation of the aforementioned systems of partial differential equations.

**Euler Equations**

In this section we will derive the Euler equations for an incompressible, ideal fluid via Chorin and Marsden’s method in [6].

Let \(D\) be a region in the two or three dimensional space that is filled with a fluid. Assume that this fluid is in motion. Our goal is to describe this motion. Let \(\vec{x} \in D\) be a point in \(D\) and consider a particle of the fluid moving through \(\vec{x}\) at time \(t\). Let \(\vec{u}(\vec{x}, t)\) denote the velocity of the particle moving through \(\vec{x}\) at time \(t\). Thus, we have for each fixed time \(t\), \(\vec{u}(\vec{x}, t)\) is a velocity vector field on \(D\). See Figure 1.
For all times $t$ we will assume that the fluid has a well-defined density $\rho(\vec{x}, t)$. So, if we consider any subregion $W$ of $D$, then the total mass for $W$ at time $t$ is given by

$$m(W, t) = \int_{W} \rho(\vec{x}, t) dV(\vec{x})$$  \hspace{1cm} (1.3)$$

where $dV$ is the volume element in the plane or space.

For the rest of our derivation of the Euler equations, we will assume enough regularity on $\vec{u}(\vec{x}, t)$ and $\rho(\vec{x}, t)$, as well as other functions to be introduced later, in order to perform the necessary calculus operations. Our derivation will also be based upon the three following basic principles:

1. mass is neither created nor destroyed (Conservation of Mass),

2. the rate of change of momentum of a portion of the fluid equals the force applied to it (Newton’s second law),

3. energy is neither created nor destroyed (Conservation of Energy).
**Conservation of Mass:** Let $W$ be a fixed subregion of $D$. In other words, $t$ is fixed. The rate of change of mass in $W$ is

$$
\frac{d}{dt}m(W,t) = \frac{d}{dt}\int_W \rho(\vec{x},t)dV(\vec{x}) = \int_W \partial_t \rho(\vec{x},t)dV(\vec{x}).
$$

Let $\partial W$ denote the boundary of $W$, assumed to be smooth, let $\vec{n}$ denote the unit outward normal defined at the points of $\partial W$, and let $dA$ denote the area element on $\partial W$. The volume flow rate across $\partial W$ per unit area is $\vec{u} \cdot \vec{n}$ and the mass flow rate per unit area is $\rho \vec{u} \cdot \vec{n}$. See Figure 2.

![Figure 1.2. Boundary of the fluid region.](image)

Thus, we have that the total mass flow rate across $\partial W$ is the following surface
Integral.

\[ \int_{\partial W} \rho \vec{u} \cdot \vec{n} \, dA. \] (1.5)

The Conservation of Mass principle can also be stated as: the rate of change of mass in \( W \) equals the rate at which mass is crossing \( \partial W \) in the inward direction, i.e.

\[ \frac{d}{dt} \int_W \rho \, dV = -\int_{\partial W} \rho \vec{u} \cdot \vec{n} \, dA. \] (1.6)

Equation (1.6) is known as the integral form of the Conservation of Mass. By the divergence theorem, we have that (1.6) becomes

\[ \int_W (\partial_t \rho + \nabla \cdot (\rho \vec{u})) \, dV = 0. \] (1.7)

Since this is to hold for all \( W \), it is equivalent to

\[ \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0. \] (1.8)

Equation (1.8) is known as the differential form of the Conservation of Mass and also known as the continuity equation.

**Balance of Momentum:** Let \( \vec{x}(t) = (x(t), y(t), z(t)) \) be the path followed by a fluid particle, so that the velocity vector field is given by

\[ \frac{d\vec{x}(t)}{dt} = \vec{u}(\vec{x}(t), t). \] (1.9)
So, we have that the acceleration of the fluid particle is given by

\[ \vec{a}(t) = \frac{d^2\vec{x}(t)}{dt^2}. \] (1.10)

By (1.9) and the chain rule, we have

\[ \vec{a}(t) = (\partial_x \vec{u})(\dot{x}(t)) + (\partial_y \vec{u})(\dot{y}(t)) + (\partial_z \vec{u})(\dot{z}(t)) + \partial_t \vec{u}. \] (1.11)

Given that

\[ \vec{u}(x, y, z, t) = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t)) \] (1.12)

we obtain

\[ \vec{a}(t) = u\partial_x \vec{u} + v\partial_y \vec{u} + w\partial_z \vec{u} + \partial_t \vec{u}. \] (1.13)

We may also write (1.13) as

\[ \vec{a}(t) = \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = \frac{D\vec{u}}{Dt}. \] (1.14)

We call \( \frac{D}{Dt} \) the material derivative.

For any continuum, forces acting on a piece of material are of two types:

1. **Forces of stress:** The piece of material is acted on by forces across its surface by the rest of the continuum.

2. **External (or body) forces:** These exert a force per unit volume on the continuum. Examples would include gravity and magnetic fields.

Now let’s give the definition of an ideal fluid.
Definition 1. An ideal fluid is a fluid with the property that for any motion of the fluid there is a function \( p(x, t) \) called the pressure such that if \( S \) is a surface in the fluid with a chosen unit normal \( \vec{n} \), the force of stress exerted across the surface \( S \) per unit area at \( x \in S \) at time \( t \) in the direction of \( \vec{n} \) is \( p(x, t)\vec{n} \). See Figure 3.

For the rest of our derivation, we will concern ourselves only with ideal fluids.

If \( W \) is a region in the fluid at a particular instant of time \( t \), the total force exerted on \( W \) by means of stresses on its boundary is

\[
\vec{S}_{\partial W} = - \int_{\partial W} p\vec{n} \, dA.
\]  

(1.15)

Here the negativity comes from \( \vec{n} \) pointing outwards. If \( \vec{e} \) is any fixed vector in space,
then we have the following.

\[
\vec{e} \cdot \vec{S}_{\partial W} = - \int_{\partial W} p \vec{e} \cdot \vec{n} dA \\
= - \int_W \nabla \cdot (pe) dV \\
= - \int_W \nabla p \cdot \vec{e} dV. \tag{1.16}
\]

Thus, we obtain

\[
\vec{S}_{\partial W} = - \int_W \nabla p \ dV. \tag{1.17}
\]

If \( \vec{b}(\vec{x}, t) \) denotes the given body force per unit mass, then the total body force is

\[
\vec{B} = \int_W \rho \vec{b} \ dV. \tag{1.18}
\]

Thus, on any piece of fluid material we have

\[
\text{force per unit volume} = -\nabla p + \rho \vec{b}.
\]

By Newton’s second law (Force = (mass) x (acceleration)), we obtain the \textit{differential form of Balance of Momentum}:

\[
\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{b}. \tag{1.19}
\]

Next, we will derive an \textit{integral form of the Balance of Momentum}. From (1.19), we
have

\[ \rho \partial_t \vec{u} = -\rho (\vec{u} \cdot \nabla) \vec{u} - \nabla p + \rho \vec{b}. \]  

(1.20)

Applying the equation of continuity (1.8), we obtain

\[ \partial_t (\rho \vec{u}) = -\nabla \cdot (\rho \vec{u}) - \rho (\vec{u} \cdot \nabla) \vec{u} - \nabla p + \rho \vec{b}. \]  

(1.21)

If \( \vec{e} \) is any fixed vector in space, we have

\[ \vec{e} \cdot \partial_t (\rho \vec{u}) = -\nabla \cdot (\rho \vec{u}) \vec{e} - \rho (\vec{u} \cdot \nabla) \vec{u} \cdot \vec{e} - \rho \vec{u} \cdot \vec{e} + \rho \vec{b} \cdot \vec{e} \]

\[ = -\nabla \cdot (p \vec{e} + \rho \vec{u} (\vec{u} \cdot \vec{e})) + \rho \vec{b} \cdot \vec{e}. \]  

(1.22)

Therefore, if \( W \) is a fixed volume in the fluid, the rate of change of momentum in the direction of \( \vec{e} \) in \( W \) is

\[ \vec{e} \cdot \frac{d}{dt} \int_W \rho \vec{u} \, dV = -\int_{\partial W} (p \vec{e} + \rho \vec{u} (\vec{u} \cdot \vec{e})) \cdot \vec{n} \, dA + \int_W \rho \vec{b} \cdot \vec{e} \, dV. \]  

(1.23)

by the Divergence Theorem. Thus, the integral form of the Balance of Momentum is the following.

\[ \frac{d}{dt} \int_W \rho \vec{u} \, dV = -\int_{\partial W} (p \vec{n} + \rho \vec{u} (\vec{u} \cdot \vec{n})) \, dA + \int_W \rho \vec{b} \, dV. \]  

(1.24)

Here, the quantity \( p \vec{n} + \rho \vec{u} (\vec{u} \cdot \vec{n}) \) is the momentum flux per unit area crossing the boundary \( \partial W \) and \( \vec{n} \) is the outward unit normal vector to \( \partial W \).

Let’s note that the above integral form came directly from the differential form
for the Balance of Momentum. However, we could have gone from the integral form of the Balance of Momentum and derived its differential form. To do this, we will introduce a few mathematical notions.

Let $D$ be our region in which the fluid is moving. Let $\vec{x} \in D$ and $\vec{\varphi}(\vec{x}, t)$ be the trajectory followed by the particle which is at the point $\vec{x}$ at time $t = 0$. We assume $\vec{\varphi}$ is smooth enough and, for a fixed time $t$, is invertible. Let $\vec{\varphi}_t$ denote the map $\vec{x} \mapsto \vec{\varphi}(\vec{x}, t)$ (here the subscripted $t$ does not imply the partial derivative with respect to our time variable) with fixed $t$. This map drives each fluid particle from its position at time $t = 0$ to its position at time $t$.

If $W$ is a sub region in $D$, then we have that $\vec{\varphi}_t(W) = W_t$ is the volume $W$ moving with the fluid. See Figure 4.

Figure 1.4. Trajectory map. [6]
The primitive integral form of the Balance of Momentum states that

$$\frac{d}{dt} \int_{W_t} \rho \vec{u} \, dV = \tilde{S}_{\partial W_t} + \int_{W_t} \rho \vec{b} \, dV. \quad (1.25)$$

In other words, the rate of change of momentum of a moving piece of fluid equals the total force acting on it, where the total force is equal to the sum of the surface stresses plus the body forces. These two forms (1.19) and (1.25), the differential form and the integral form of the Balance of Momentum, are equivalent. To prove this, we will make the following change of variables.

$$\frac{d}{dt} \int_{W_t} \rho \vec{u} \, dV = \frac{d}{dt} \int_{W} (\rho \vec{u})(\vec{\varphi}(\vec{x}, t), t) J(\vec{x}, t) \, dV, \quad (1.26)$$

where $J(\vec{x}, t)$ is the Jacobian determinant of the map $\vec{\varphi}_t$. Before we proceed, we will need the following lemma.

**Lemma 1.** $\partial_t J(\vec{x}, t) = J(\vec{x}, t) \nabla \cdot \vec{u}(\vec{\varphi}(\vec{x}, t), t)$.

**Proof.** Let us write $\vec{\varphi}$ as $\vec{\varphi}(\vec{x}, t) = \vec{\varphi}(\xi(\vec{x}, t), \eta(\vec{x}, t), \zeta(\vec{x}, t))$. We also observe that

$$\begin{cases}
\partial_t \vec{\varphi}(\vec{x}, t) = \vec{u}(\vec{\varphi}(\vec{x}, t), t) \\
\vec{\varphi}(\vec{x}, 0) = \vec{x}
\end{cases} \quad (1.27)$$

by the definition of the velocity field of the fluid. By utilizing the definiton of the
Jacobian of \( \vec{\varphi} \), we differentiate with respect to our time variable and achieve

\[
\partial_t J(\vec{x}, t) = \begin{vmatrix}
\partial_t \partial_x \xi & \partial_x \eta & \partial_x \zeta \\
\partial_t \partial_y \xi & \partial_y \eta & \partial_y \zeta \\
\partial_t \partial_z \xi & \partial_z \eta & \partial_z \zeta
\end{vmatrix}
\begin{vmatrix}
\partial_x \xi & \partial_x \eta & \partial_x \zeta \\
\partial_y \xi & \partial_y \eta & \partial_y \zeta \\
\partial_z \xi & \partial_z \eta & \partial_z \zeta
\end{vmatrix}
\begin{vmatrix}
\partial_x \xi & \partial_x \eta & \partial_x \zeta \\
\partial_y \xi & \partial_y \eta & \partial_y \zeta \\
\partial_z \xi & \partial_z \eta & \partial_z \zeta
\end{vmatrix}
\] (1.28)

Now, we have from (1.27)

\[
\begin{cases}
\partial_t \partial_x \xi = \partial_x \partial_t \xi = \partial_x u(\vec{\varphi}(\vec{x}, t), t) \\
\partial_t \partial_y \xi = \partial_y \partial_t \xi = \partial_y u(\vec{\varphi}(\vec{x}, t), t) \\
\quad \vdots \\
\partial_t \partial_z \zeta = \partial_z \partial_t \zeta = \partial_z w(\vec{\varphi}(\vec{x}, t), t),
\end{cases}
\] (1.29)

where \( u, v, \) and \( w \) are the components of \( \vec{u} \). By the chain rule, we have

\[
\begin{cases}
\partial_x u(\vec{\varphi}(\vec{x}, t), t) = \partial_x u \partial_x \xi + \partial_y u \partial_x \eta + \partial_z u \partial_x \zeta \\
\partial_y u(\vec{\varphi}(\vec{x}, t), t) = \partial_x u \partial_y \xi + \partial_y u \partial_y \eta + \partial_z u \partial_y \zeta \\
\quad \vdots \\
\partial_z w(\vec{\varphi}(\vec{x}, t), t) = \partial_x w \partial_z \xi + \partial_y w \partial_z \eta + \partial_z w \partial_z \zeta.
\end{cases}
\] (1.30)

Substituting relations (1.29) and (1.30) into (1.28), we obtain

\[
\partial_x uJ + \partial_y vJ + \partial_z wJ = \nabla \cdot \vec{u} J.
\] (1.31)
This concludes the proof of our lemma.

Utilizing Lemma 1 we have the following.

\[
\frac{d}{dt} \int_{W_t} \rho \vec{u} \, dV = \int_{W_t} \left( \left( \frac{D}{Dt} \rho \vec{u} \right)(\vec{x}, t) + \nabla \cdot \vec{u}(\rho \vec{u})(\vec{x}, t) \right) J(\vec{x}, t) \, dV \\
= \int_{W_t} \left( \frac{D}{Dt} (\rho \vec{u}) + (\rho \nabla \cdot \vec{u}) \vec{u} \right) \, dV. \tag{1.32}
\]

Applying the differential form of the Conservation of Mass (1.8) where we have

\[
\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = \partial_t \rho + \nabla \cdot (\rho \vec{u}) = 0, \tag{1.33}
\]

we obtain

\[
\frac{d}{dt} \int_{W_t} \rho \vec{u} \, dV = \int_{W_t} \rho \frac{D\vec{u}}{Dt} \, dV. \tag{1.34}
\]

In fact, this gives us the following theorem.

**Transport Theorem:** For any function \( f \) of \( \vec{x} \) and \( t \), we have

\[
\frac{d}{dt} \int_{W_t} \rho f \, dV = \int_{W_t} \rho \frac{Df}{Dt} \, dV.
\]

If \( W \), and hence \( W_t \), is arbitrary, then the primitive integral form of the Balance of Momentum is equivalent to

\[
\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho \vec{b}, \tag{1.35}
\]

which is the differential form of the Balance of Momentum. Hence, we have that (1.19), (1.24), (1.25) are all equivalent.

We now take note that our Lemma 1 aids us in understanding incompressibility.
We call a flow *incompressible* if for any fluid subregion $W$,

$$\text{volume}(W_t) = \int_{W_t} dV = \text{constant in } t. \quad (1.36)$$

Thus, we have that

$$0 = \frac{d}{dt} \int_{W_t} dV = \frac{d}{dt} \int_{W} J \, dV = \int_{W} \nabla \cdot \vec{u} J \, dV = \int_{W_t} \nabla \cdot \vec{u} \, dV \quad (1.37)$$

for all moving regions $W_t$.

Thus the following are equivalent.

1. The fluid is incompressible.

2. $\nabla \cdot \vec{u} = 0$.

3. $J \equiv 1$.

From the differential form of the Conservation of Mass (1.8), we find that a fluid is incompressible if and only if $\frac{D\rho}{Dt} = 0$. Which would imply for the density of the fluid to be constant in space and time.

We shall now solve equation (1.8). If we let $f = 1$ in our Transport Theorem, then we have

$$\frac{d}{dt} \int_{W_t} \rho \, dV = 0 \quad \text{(Conservation of Mass)}, \quad (1.38)$$

and thus,

$$\int_{W_t} \rho(\vec{x}, t) \, dV = \int_{W_0} \rho(\vec{x}, 0) \, dV. \quad (1.39)$$

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By a change of variables we have

\[
\int_{W_0} \rho(\vec{\varphi}(\vec{x}, t), t) J(\vec{x}, t) \, dV = \int_{W_0} \rho(\vec{x}, 0) \, dV. \tag{1.40}
\]

Since \( W_0 \) is arbitrary, we obtain

\[
\rho(\vec{\varphi}(\vec{x}, t), t) J(\vec{x}, t) = \rho(\vec{x}, 0), \tag{1.41}
\]

which is another form of the Conservation of Mass.

**Conservation of Energy:** At this point, let’s recap what we have derived thus far.

1. \( \rho \frac{D\vec{u}}{Dt} = -\nabla p - \rho \vec{b} \). (Ideal Fluid)

2. \( \frac{D\rho}{Dt} + \rho \nabla \cdot \vec{u} = 0 \). (Conservation of Mass)

This gives us four equations if we are working in three dimensions, however, we have five functions \( \vec{u}, \rho \) and \( p \). This should lead us to suspect that there is yet another equation to develop. This equation will come from another conservation law, namely, the Conservation of Energy.

For a fluid moving in a domain \( D \), with velocity vector field \( \vec{u} \), the kinetic energy of the fluid is given by

\[
E_{\text{kinetic}} = \frac{1}{2} \int_D \rho |\vec{u}|^2 \, dV, \tag{1.42}
\]

where \( |\vec{u}|^2 = (u^2 + v^2 + w^2) \) is the square length of our velocity vector function \( \vec{u} \). We
will assume that the total energy of the fluid can be written as

\[ E_{\text{total}} = E_{\text{kinetic}} + E_{\text{internal}}. \]

Here, the internal energy refers to the energy that is only noticeable at the microscopic level. From here, we will assume that the internal energy is negligible as well as the external forces \( \vec{b} \). However, if the external forces \( \vec{b} \) are derived from a potential \( \nu \), then the external energy

\[ E_{\text{external}} = \int_D \rho \nu \, dV \]

would need to be added to the total energy. From the Conservation of Energy, we have that

\[ \frac{d}{dt} \left( \frac{1}{2} \int_D \rho |\vec{u}|^2 \, dV \right) = 0. \]  

(1.43)

By the Transport Theorem, this is equivalent to

\[ \int_D \rho \vec{u} \cdot \frac{D\vec{u}}{Dt} \, dV = 0. \]  

(1.44)

From our derived equation of the flow of an ideal fluid, we have from (1.44) that

\[ -\int_D \vec{u} \cdot \nabla p \, dV = 0. \]  

(1.45)

Integrating (1.45) by parts, assuming \( \vec{u} \cdot \vec{n} = 0 \) where \( \vec{n} \) is the unit normal vector on \( \partial D \), we obtain

\[ \frac{d}{dt} (E_{\text{kinetic}}) = \int_D (\nabla \cdot \vec{u}) p \, dV = 0. \]  

(1.46)
This implies that either $\nabla \cdot \vec{u} = 0$ or $p = 0$. The case where $\nabla \cdot \vec{u} = 0$ is that of an incompressible fluid which we discussed earlier. The case where $p = 0$ is possible, but not an interesting case.

Now we have the Euler equations for an incompressible, ideal fluid to be

$$\begin{cases} 
\rho \frac{D\vec{u}}{Dt} = -\nabla p \\
\frac{D\rho}{Dt} = 0 \\
\nabla \cdot \vec{u} = 0 \\
\text{boundary condition: } \vec{u} \cdot \vec{n} = 0 \text{ on } \partial D.
\end{cases}$$

(1.47)

1.2 DERIVATION OF CH2

In this section, we work our way towards the derivation of the CH2 system and find that the set of PDEs stems from the Green-Naghdi (GN) equations. The GN system was derived directly from the Euler equations in the 1970’s in the context of shallow water wave theory. We will now take the Euler equations that we have derived from Newton’s Second Law above and derive the GN system. Afterwards, we will follow the methods of Constantin and Ivanov [9] and asymptotically expand the GN system to obtain the CH2 system.

Green-Naghdi Equations

In this section, we follow R.S. Johnson’s approach from [37]. Let us now consider an inviscid fluid of constant depth, which is stationary in its undisturbed state. Here we also neglect surface tension. We have that the governing equations for this scenario are the Euler equations with appropriate prescribed boundary conditions. Further-
more, we non-dimensionalize the problem using the vertical height length scale $h$, wavelength $\lambda$, and amplitude of the wave $a$. See Figure 5.

![Figure 1.5. Shallow water waves.](image)

The appropriate non-dimensionalization for the horizontal velocity component is $\sqrt{gh}$, where $g$ is gravity, with the corresponding time $\lambda/\sqrt{gh}$. The non-dimensional equations will now contain two parameters $\varepsilon = a/h$, a.k.a. the amplitude parameter and $\delta = h/\lambda$ the shallowness parameter. Writing the surface as $z = 1 + \varepsilon \eta(x, t)$ and letting $p$ be our pressure, we have our governing equations to be
\[ u_t + \varepsilon (u u_x + w u_z) = -p_x, \] (1.48)
\[ \delta^2 \{ w_t + \varepsilon (u w_x + w w_z) \} = -p_z, \] (1.49)
\[ u_x + w_z = 0, \] (1.50)

with
\[ p = \eta, \quad w = \eta_t + \varepsilon u \eta_x \] on \( z = 1 + \varepsilon \eta, \) (1.51)

and
\[ w = 0 \] on \( z = 0. \) (1.52)

To obtain the Green-Naghdi (GN) equations, we assume \( u \) is not a function of \( z. \)

From (1.50), we have that
\[ w = -z u_x. \]

From equation (1.49), we now have
\[ p = \eta - \frac{1}{2} \delta^2 \{(1 + \varepsilon \eta)^2 - z^2\} (u_{xt} + \varepsilon u u_x - \varepsilon u_x^2), \]

which satisfies the pressure boundary condition (1.51) at the surface. Plugging the above into equation (1.48), we obtain
\[ u_t + \varepsilon u u_x + \eta_x = \frac{\delta^2/3}{(1 + \varepsilon \eta)} [(1 + \varepsilon \eta)^3 (u_{xt} + \varepsilon u u_x - \varepsilon u_x^2)]_x. \] (1.53)

The second equation relating \( \eta \) and \( u \) is given by integrating over \( z \) on (1.50). Thus
we have
\[ \eta_t + [u(1 + \varepsilon \eta)]_x = 0. \] (1.54)

Equations (1.53) and (1.54) are the GN equations for one-dimensional wave motion over a flat bed.

**CH2 System**

Using the approach given by Constantin and Ivanov [9], we obtain the 2 component Camassa-Holm system in the following manner. First, we start from the GN equations
\[
\begin{align*}
    u_t + \varepsilon uu_x + \eta_x &= \frac{\delta^2/3}{(1 + \varepsilon \eta)^3}(1 + \varepsilon \eta)^3(u_{xt} + \varepsilon uu_{xx} - \varepsilon u_x^2)_x, \\
    \eta_t + [u(1 + \varepsilon \eta)]_x &= 0.
\end{align*}
\]

The leading order expansion with respect to the parameters \(\varepsilon\) and \(\delta^2\) gives us the system
\[
\begin{align*}
    
    \left( u - \frac{\delta^2}{3} u_{xx} \right)_t + \varepsilon uu_x + \eta_x &= 0, \\
    \eta_t + [u(1 + \varepsilon \eta)]_x &= 0.
\end{align*}
\] (1.55) (1.56)

Now, let us define
\[ \rho = 1 + \frac{1}{2} \varepsilon \eta - \frac{1}{8} \varepsilon^2 (u^2 + \eta^2). \]

Expanding \(\rho^2\) and retaining only \(O(\varepsilon)\), we have
\[ \rho^2 = 1 + \varepsilon \eta - \frac{1}{4} \varepsilon^2 u^2. \]
Using the above, we may write (1.55) as

\[
(u - \frac{\delta^2}{3} u_{xx})_t + \frac{3}{2} \varepsilon uu_x + \frac{1}{\varepsilon} (\rho^2)_x = 0. \tag{1.57}
\]

Next, we use the fact that \(u_t \approx -\eta_x\) and \(\eta_t \approx -u_x\) to obtain

\[
\rho_t = \frac{1}{2} \varepsilon \eta_t + \frac{1}{4} \varepsilon^2 (\eta u)_x.
\]

With the above expression for \(\rho_t\) and \(\rho \approx 1 + \varepsilon \eta\), we have that (1.56) may be written as

\[
\rho_t + \frac{\varepsilon}{2} (\rho u)_x = 0. \tag{1.58}
\]

Rescaling \(u \rightarrow \frac{2}{\varepsilon} u, x \rightarrow \frac{\delta}{\sqrt{3}} x, t \rightarrow \frac{\delta}{\sqrt{3}} t\) in (1.57) and (1.58) gives us the 2 component Camassa-Holm equation

\[
\begin{cases}
  u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + \sigma \rho \rho_x = 0 \\
  \rho_t + (u \rho)_x = 0
\end{cases}
\tag{1.59}
\]

with \(\sigma = 1\). The case where \(\sigma = -1\) corresponds to when the direction of gravitational acceleration points upward.
CHAPTER 2
INTRODUCTION AND RESULTS

In this thesis, we consider the periodic and non-periodic Cauchy problem of the 2-component Camassa-Holm (CH2) system

\[
\begin{aligned}
\partial_t u + u \partial_x u + (1 - \partial_x^2)^{-1} \partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 + \frac{\sigma}{2} \rho^2 \right] &= 0 \\
\partial_t \rho + \partial_x (u \rho) &= 0 \\
u(x, 0) &= u_0(x) \\
\rho(x, 0) &= \rho_0(x),
\end{aligned}
\]

(2.1)

where \( \sigma = 1 \) or \( \sigma = -1 \). Here \( u(x, t) \) represents the fluid velocity at a time \( t \) in the spatial direction \( x \) and \( \rho(x, t) \) denotes the horizontal deviation of the surface from equilibrium. In 2008, Constantin and Ivanov [9] demonstrated how the system (2.1) is derived from shallow water wave theory through asymptotic expansions of the Green-Naghdi equations. Furthermore, it was later discovered that the 2-component system is completely integrable and has a bi-Hamiltonian structure as shown in Falqui [16] and Shabat and Alonso [50]. Consideration of a more general coupled Camassa-Holm type system is discussed in Zhu [58].
In recent years, well-posedness results on the above system have been established both on the circle and the real line in Sobolev spaces $H^s \times H^{s-1}$ with $s \geq 2$, as found in Escher, Lechtenfeld and Yin [14], and in Hu and Yin [34] where they utilized Kato’s semigroup theory, which can also be found in a recent paper by Ni and Zhou [48]. Blow-up phenomena was subsequently examined and the system was shown to generate smooth solutions that develop singularities in finite time in the form of wave breaking, i.e. the solution $u(x,t)$ remains bounded while its first derivative with respect to the spatial direction $x$ becomes unbounded at some time $T > 0$, as demonstrated in [9], [14], [34], and in Guo and Zhou [25]. The system has also been shown to have the following conservation laws

$$\int_{\mathbb{R}} (u^2 + (\partial_x u)^2 + \sigma \rho^2) \, dx \quad \text{and} \quad \int_{\mathbb{R}} (u^3 + u(\partial_x u)^2 + \sigma u \rho^2) \, dx,$$

which Guo [23] utilized in demonstrating the aforementioned property of wave breaking for the system (2.1). Guo and Zhu [26] also showed that the modified two-component Camassa-Holm system, with $\rho = (1 - \partial_x^2)(\bar{\rho} - \rho_0)$ where $\bar{\rho}_0$ is a constant, also exhibits the wave breaking property. Furthermore, existence of global solutions along with the aforementioned blow-up behavior has been investigated in Guan and Yin [19], Gui and Liu [21] and Yuen [57].

In 2009, David Henry [27] showed that solutions to the two-component Camassa-Holm system have an infinite speed propagation. The main result (Theorem 3.6) purports that no matter what the profile of the compactly supported initial datum $u_0(x)$ and $\rho_0(x)$ (whether they are positive or negative), for any $t > 0$ in its lifespan, the solution $u(x,t)$ is positive at infinity and negative at negative infinity. This
idea first appeared in the work of Himonas, Misiolek, Ponce, and Zhou [33]. Guo and Ni [24] also investigated the infinite propagation speed in the sense that the corresponding solution does not have compact spatial support for $t > 0$ though the initial data belongs to $C^\infty_0(\mathbb{R})$. Furthermore, in relation to the profiles of the initial datum to our system (2.1), Guo [22] demonstrated that the corresponding solution to initial data with algebraic decay at infinity will retain this property at infinity in its lifespan. Jin and Guo [36] then extended this argument to the modified two-component CH system where $\rho = (1 - \partial_x^2)(\bar{\rho} - \bar{\rho}_0)$ and $\bar{\rho}_0$ is a constant.

When $\rho = 0$ in system (1.1) we have the celebrated Camassa-Holm equation

\[(1 - \partial_x^2)\partial_tu + 3u\partial_xu - 2\partial_xu\partial_x^2u - u\partial_x^3u = 0,\]  

(2.2)

which was derived physically by Camassa and Holm [2] as a nonlinear weakly dispersive wave equation that models the propagation of shallow water waves over a flat surface. The CH equation was also derived in a purely geometric context as the equation of geodesics of the Sobolev $H^1$ metric on the group of diffeomorphisms of the circle as shown in Misiolek [45]. Equation (2.2) first appeared in the works of Fokas and Fuchssteiner [17] pertaining to their research on hereditary symmetries of integrable PDEs. Due to the interesting and fascinating properties that the Camassa-Holm equation exhibits, the above equation has drawn a profound amount of attention among researchers during previous years.

From past research on the CH equation, we know that it is completely integrable [2] and exhibits a bi-Hamiltonian structure as demonstrated in Liu and Zhang [42].
Furthermore, equation (2.2) has an infinite number of conservation laws and generates solitary wave solutions which are solitons with stable profiles. One of which is the existence of peaked solitary waves or “peakons” which are of the form $u(x, t) = ce^{-|x-ct|}$ where $c \neq 0$ is the finite speed \cite{2}. These peakons are smooth except at the crest of the solution where there is a discontinuity in the first derivative.

Another interesting fact about the Camassa-Holm equation is that it generates smooth solutions that develop singularities in finite time in the form of wave breaking. The examination of blow-up phenomena pertaining to (2.2) relies on a specific technique in which one considers a family of diffeomorphisms of the real line. In 1998 McKean \cite{44} gave a most delicate proof of the wave breaking property.

From the CH-equation we take its short wave limit and achieve the Hunter-Saxton (HS) \cite{35} equation
\[
\partial_t \partial_x^2 u + 2 \partial_x u \partial_x^2 u + u \partial_x^3 u = 0,
\]
which models the propagation of waves in a massive director field of a nematic liquid crystal, with the orientation of the molecules described by the field of unit vectors $n = (\cos(u(x, t)), \sin(u(x, t)))$, where $x$ is the spacial variable in a reference frame moving with the linearized wave velocity, and $t$ is a slow time variable. If we take the HS equation and couple it with the CH equation, we find that they generate many integrable multicomponent generalizations. In this paper, we examine only one particular integrable multicomponent system which is the 2-component CH system (2.1). For other works related to ours about Camassa-Holm type equations and systems and other various shallow water wave equations we refer the reader to \cite{3, 4, 7, 8, 11, 12, 18, 20, 38, 39, 41, 43, 47, 49, 51, 53} and the references therein.
Now we state our first result, which is for data in Sobolev spaces on both the line and the circle. For this, we use the notation

\[ H^s \times H^{s-1}(\mathbb{R} \text{ or } T) \equiv H^s \times H^{s-1}. \]

When we work on \( T \), however, we shall use \( H^s \times H^{s-1}(T) \) and when on \( \mathbb{R} \) we use \( H^s \times H^{s-1}(\mathbb{R}) \).

**Theorem 1.** Given \( z_0 = (u_0, \rho_0) \in H^s \times H^{s-1}, \ s > 5/2, \) there exists a maximal \( T = T(\|z_0\|_{H^s \times H^{s-1}}) > 0 \) and a corresponding unique solution \( z = (u, \rho) \) to (2.1) such that

\[ z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}). \]

Moreover, the solution depends continuously upon the initial data, i.e. the mapping

\[ z_0 \to z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \]

is continuous and we have the estimate

\[ \|(u(t), \rho(t))\|_{H^s \times H^{s-1}} \leq 2\|(u_0, \rho_0)\|_{H^s \times H^{s-1}}, \text{ for } 0 \leq t \leq T \leq \frac{1}{2c_s \cdot (\|(u_0, \rho_0)\|_s)}, \]

(2.3)

where \( c_s > 0 \) is a constant depending on \( s \).

Here we should note that the norm \( \| \cdot \|_{H^s \times H^{s-1}} \) is defined and abbreviated \( \| \cdot \|_s \).
as follows
\[ \|(u, \rho)\|_s = \|(u, \rho)\|_{H^s \times H^{s-1}} \leq \|u\|_{H^s} + \|\rho\|_{H^{s-1}}. \]

We also may, in some cases, drop the $\mathbb{T}$ (or $\mathbb{R}$) in parentheses since it is understood for all our results to be acquired in Sobolev spaces over the torus (resp. real line) for Theorem 1.

Our second result, which is motivated by the works of Himonas and Holliman [29], Himonas and Kenig [30], Himonas, Kenig and Misiolek [31], and Himonas, Misiolek and Ponce [32] demonstrates the continuous dependence on the initial data is sharp.

**Theorem 2.** If $s > 5/2$ then the data-to-solution map for the 2-component Camassa-Holm system defined by the Cauchy problem (2.1) is not uniformly continuous from any bounded subset of $H^s \times H^{s-1}$ into $C([0, T]; H^s \times H^{s-1})$.

To demonstrate this sharpness of continuity, sequences of approximate solutions are constructed. Then the actual solutions are found by solving the Cauchy problem with initial data given by the approximate solutions at time $t = 0$. The error produced by solving the Cauchy problem using approximate solutions is shown to be inconsequential.

Since the data-to-solution map is not uniformly continuous, the local well-posedness result in Theorem 1 cannot be shown through the use of a fixed point theorem for contraction mappings. Rather, the proof depends on a refined well-posedness result which includes a solution size estimate. Then, based on a Galerkin-type approximation, we can extract the above local well-posedness result for the 2-component CH-system.
Following our proofs of Theorem 1 and 2 we shall establish similar proofs for the non-periodic case (on the real line) in sections seven and eight and then expand our range of continuity properties by considering weaker topologies on $H^s \times H^{s-1}$. Although the data-to-solution map is not uniformly continuous on $H^s \times H^{s-1}$, we will show that the map is Hölder continuous if we choose a properly weakened topology. The proof is based on energy estimates used to demonstrate uniqueness, which depends on a commutator estimate from Taylor[32]. To be more precise, we subdivide the set $\mathcal{I} = \{(s,r) : s > 5/2, 0 \leq r < s\}$ of Sobolev exponents into two regions as follows

$$\mathcal{I}_1 = \{(s,r) : s > 5/2, 1/2 < r \leq s - 1, \ r \geq 2 - s\}, \quad (2.4)$$

$$\mathcal{I}_2 = \{(s,r) : s > 5/2, \ s - 1 < r < s\}. \quad (2.5)$$

and define the function $\alpha = \alpha(s,r)$ on $\mathcal{I}$ as

$$\alpha = \alpha(s,r) = \begin{cases} 1, & \text{if } (s,r) \in \mathcal{I}_1 \\ s - r, & \text{if } (s,r) \in \mathcal{I}_2. \end{cases} \quad (2.6)$$

Let $B(0, \lambda) \subset H^s \times H^{s-1}$ denote the ball of radius $\lambda$ in $H^s \times H^{s-1}$; i.e.

$$B(0, \lambda) = \{(u,\rho) \in H^s \times H^{s-1} : \|(u,\rho)\|_s < \lambda\}. \quad (2.7)$$

Given (2.4)-(2.7) we will prove the following theorem at the end of the paper.

**Theorem 3.** If $s > 5/2$ and $0 \leq r < s$, then the data-to-solution map of (2.1) is
Hölder continuous with the exponent $\alpha = \alpha(s,r)$ defined in (1.11) as a map from $B(0,\lambda) \subset H^s \times H^{s-1}$, with $H^r \times H^{r-1}$ norm, to $C([0,T]; H^r \times H^{r-1})$; i.e. there exist positive constants $T = T(s,r,\lambda)$ and $c = c(s,r,\lambda)$ such that

$$\|(u(t) - w(t), \rho(t) - \phi(t))\|_{C([0,T]; H^r \times H^{r-1})} \leq c\|(u(0) - w(0), \rho(0) - \phi(0))\|_r^\alpha \quad (2.8)$$

holds for all $(u(0), \rho(0)), (w(0), \phi(0)) \in B(0,\lambda)$, where $(u(t), \rho(t))$ and $(w(t), \phi(t))$ are solutions of (1.3) with initial data $(u(0), \rho(0))$ and $(w(0), \phi(0))$ respectively.

The paper is structured as follows. In chapter three, we prove Theorem 1 on the torus. We follow up with the proof of Theorem 2 on the torus in chapter four. Then in chapter 5, we prove Theorem 1 on the real line and follow up in chapter six with the proof of Theorem 2 on the real line. Finally, in chapter seven we prove Theorem 3 and then conclude with some numerics in chapter eight.
CHAPTER 3

WELL-POSEDNESS ON THE TORUS

3.1 PRELIMINARY ESTIMATES AND LIFESPAN

For the sake of simplicity from this section till the end of section six we shall use the notation

\[ H^s(\mathbb{T} \text{ or } \mathbb{R}) = H^s. \]

To study the analytic properties of (2.1) we write the equation in its non-local form

\[
\begin{aligned}
\partial_t u + u\partial_x u + (1 - \partial_x^2)^{-1}\partial_x \left[ u^2 + \frac{1}{2} (\partial_x u)^2 + \frac{\sigma}{2} \rho^2 \right] &= 0, \\
\partial_t \rho + \partial_x (u\rho) &= 0.
\end{aligned}
\]

(3.1)

For brevity, will write the 2-component CH system as follows

\[
\begin{aligned}
\partial_t u + u\partial_x u + F(u, \rho) &= 0, \quad x \in \mathbb{T} \\
\partial_t \rho + \partial_x (u\rho) &= 0
\end{aligned}
\]

where

\[ F(u, \rho) = F_1(u, \rho) + F_2(u, \rho) + F_3(u, \rho), \]
with

\[ F_1(u, \rho) = D^{-2} \partial_x (u^2), \]
\[ F_2(u, \rho) = \frac{1}{2} D^{-2} \partial_x [(\partial_x u)^2], \]
\[ F_3(u, \rho) = \frac{\sigma}{2} D^{-2} \partial_x (\rho^2). \]  

(3.2)

For any real number \( s \) the pseudo-differential operator \( D^s \) is defined by the formula

\[ \widehat{D^s f}(\xi) = (1 + \xi^2)^{s/2} \hat{f}(\xi), \]

(3.3)

where \( f \in C_0^\infty(\mathbb{R}) \). In particular, \( D^{-2} \) is of order 2 satisfying the estimate

\[ \| D^{-2} f \|_{H^r} \leq \| f \|_{H^{r-2}}. \]

(3.4)

We will also use the estimate

\[ \| \partial_x D^{-2} f \|_{H^r} \leq \| f \|_{H^{r-1}}. \]

(3.5)

Finally, we shall use an estimate for the Sobolev norm of a product. For any \( s > 0 \) there is a \( c_s > 0 \) such that

\[ \| fg \|_{H^s} \leq c_s [ \| f \|_{H^s} \| g \|_{L^\infty} + \| f \|_{L^\infty} \| g \|_{H^s}]. \]

(3.6)
When $s > 1/2$, we have by Sobolev Theorem the algebra property

$$\|fg\|_{H^s} \leq c_s \|f\|_{H^s} \|g\|_{H^s}.$$  \hfill (3.7)

Now we will give an estimate for our non-local term $F(u, \rho)$.

**Lemma 2.** For $s > 5/2$ we have $F : H^s \times H^{s-1} \to H^s$ and we have the estimate

$$\|F(u, \rho)\|_{H^s} \leq c_s \left[ \|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \right].$$ \hfill (3.8)

where $c_s$ is a constant that depends on $s$.

**Proof.** We have by the triangle inequality that

$$\|F(u, \rho)\|_{H^s} \leq \|F_1(u, \rho)\|_{H^s} + \|F_2(u, \rho)\|_{H^s} + \|F_3(u, \rho)\|_{H^s}. \hfill (3.9)$$

For bounding these terms, we will use properties (3.4)-(3.7). This will yield

$$\|F_1(u, \rho)\|_{H^s} = \|D^{-2}\partial_x (u^2)\|_{H^s} \leq \|\frac{k_1}{2} \partial_x (u^2)\|_{H^{s-2}} \leq \|u^2\|_{H^{s-1}} \lesssim \|u\|_{H^s}^2,$$

$$\|F_2(u, \rho)\|_{H^s} = \|\frac{1}{2} D^{-2}(\partial_x u)^2\|_{H^s} \lesssim \|\partial_x u\|_{H^{s-1}} \lesssim \|u\|_{H^s}^2,$$

$$\|F_3(u, \rho)\|_{H^s} = \|\frac{\sigma}{2} D^{-2}\partial_x (\rho^2)\|_{H^s} \lesssim \|\partial_x (\rho^2)\|_{H^{s-2}} \lesssim \|\rho^2\|_{H^{s-1}} \lesssim \|\rho\|_{H^{s-1}}^2 \hfill (3.10)$$

Combining these terms, we obtain the desired inequality (3.8) for our nonlocal term $F(u, \rho)$. 

\[\square\]
**Lemma 3.** Let $1/2 < \gamma < 1$. Then

$$\|fg\|_{H^{\gamma-1}} \leq c_\gamma \|f\|_{H^{\gamma}} \|g\|_{H^{\gamma-1}}$$

We may now proceed in estimating $F(u, \rho) - F(w, \phi)$ in Sobolev spaces with exponent $s > 1/2$.

**Lemma 4.** For any $(u, \rho)$ and $(w, \phi)$ in $H^{\gamma} \times H^{\gamma-1}$, we have the estimates for $\gamma \in (1/2, 1)$ and $\gamma \in (3/2, \infty)$ respectively

$$\|F(u, \rho) - F(w, \phi)\|_{H^{\gamma}} \lesssim (\|u\|_{H^{\gamma+1}} + \|w\|_{H^{\gamma+1}}) \|u - w\|_{H^{\gamma}} + (\|\rho\|_{H^{\gamma}} + \|\phi\|_{H^{\gamma}}) \|\rho - \phi\|_{H^{\gamma-1}},$$

(3.11)

$$\|F(u, \rho) - F(w, \phi)\|_{H^{\gamma}} \lesssim (\|u\|_{H^{\gamma}} + \|w\|_{H^{\gamma}}) \|u - w\|_{H^{\gamma}} + (\|\rho\|_{H^{\gamma-1}} + \|\phi\|_{H^{\gamma-1}}) \|\rho - \phi\|_{H^{\gamma-1}}.$$

(3.12)

**Proof.** Let’s assume that $1/2 < s < 1$. We will decompose $F(u, \rho) - F(w, \phi)$ by using (3.2). We have for the first term

$$\|F_1(u, \rho) - F_1(w, \phi)\|_{H^{s}} = \|D^{-2} \partial_x [u^2 - w^2]\|_{H^{s}}$$

$$\leq (\|u - w\|_{H^{s+1}} + \|w\|_{H^{s+1}}) \|u - w\|_{H^{s}}.$$
For the second term we use Lemma 2 and obtain

\[ \left\| F_2(u, \rho) - F_2(w, \phi) \right\|_{H^s} = \left\| \frac{1}{2} D^{-2} \partial_x \left[ (\partial_x u)^2 - (\partial_x w)^2 \right] \right\|_{H^s} \]

\[ \lesssim \left\| \partial_x (u + w) \partial_x (u - w) \right\|_{H^{s-1}} \]

\[ \lesssim \left\| \partial_x (u + w) \right\|_{H^s} \left\| \partial_x (u - w) \right\|_{H^{s-1}} \]

\[ \lesssim \left( \left\| u \right\|_{H^{s+1}} + \left\| w \right\|_{H^{s+1}} \right) \left\| u - w \right\|_{H^s}. \]

Finally, for the third term we have

\[ \left\| F_3(u, \rho) - F_3(w, \phi) \right\|_{H^s} = \left\| \frac{\sigma}{2} D^{-2} \partial_x (\rho^2 - \phi^2) \right\|_{H^s} \]

\[ \lesssim \left\| (\rho + \phi)(\rho - \phi) \right\|_{H^{s-1}} \]

\[ \lesssim \left( \left\| \rho \right\|_{H^s} + \left\| \phi \right\|_{H^s} \right) \left\| \rho - \phi \right\|_{H^{s-1}} \]

After combining all the above terms, we have (3.11). The proof of (3.12) is just a simple application of the algebra property (3.7). This concludes the proof for Lemma 4.

\[ \square \]

The Mollified I.V.P. Now we wish to use the following ODE Theorem.

**Theorem 4.** Let \( Y \) be a Banach space, \( X \subset Y \) be an open subset, \( I \subset \mathbb{R} \), and \( F : I \times X \rightarrow Y \) a continuously differentiable map. Then for any \( t_0 \in I \) and \( u_0 \in X \) there exists an open interval \( J \subset I \) containing \( t_0 \) and a unique differentiable mapping
$u : J \times X \rightarrow Y$ solving the initial value problem

$$\frac{du}{dt} = F(t, u(t)), \quad u(t_0) = u_0 \quad (3.13)$$

for all $t \in J$.

Since we cannot use this theorem with (3.1), we mollify the equation to obtain such a differentiable map $F$ as mentioned in the previous theorem and thus acquire a solution through methods of ODEs.

So with the above estimate now in place, we turn our attention to the mollified version of the Cauchy Problem (3.1). We take the mollified i.v.p. to be

$$\begin{cases}
\partial_t u + J_\varepsilon [J_\varepsilon u \partial_x J_\varepsilon u] + F(u, \rho) = 0, \quad x \in \mathbb{T} \\
\partial_t \rho + J_\varepsilon [J_\varepsilon u \partial_x J_\varepsilon \rho] + J_\varepsilon [J_\varepsilon \rho \partial_x J_\varepsilon u] = 0
\end{cases} \quad (3.14)$$

where for each $\varepsilon \in (0, 1]$ the operator $J_\varepsilon$ is the Friedrichs mollifier defined by

$$J_\varepsilon f \doteq j_\varepsilon * f. \quad (3.15)$$

The function $j_\varepsilon$ is defined by first fixing a Schwartz function $j(x) \in \mathcal{S}(\mathbb{R})$ and then define the periodic functions $j_\varepsilon$ by

$$j_\varepsilon(x) \doteq \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \hat{j}(\varepsilon n) e^{i\pi x}$$

satisfying $0 \leq \hat{j}(\xi) \leq 1$ for all $\xi \in \mathbb{R}$, and $\hat{j}(\xi) = 1$ for all $\xi \in [-1, 1]$.  

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We can also define $G_\varepsilon(u, \rho) : H^s \times H^{s-1} \to H^s$ and $E(u, \rho) : H^s \times H^{s-1} \to H^{s-1}$ to be the functions

$$
G_\varepsilon(u, \rho) = -J_\varepsilon [J_\varepsilon u \partial_x J_\varepsilon u] - F(u, \rho)
$$

$$
E_\varepsilon(u, \rho) = -J_\varepsilon [J_\varepsilon u \partial_x J_\varepsilon \rho] - J_\varepsilon [J_\varepsilon \rho \partial_x J_\varepsilon u].
$$

Then we have that

$$
\partial_u G_\varepsilon(u, \rho) h = -J_\varepsilon [J_\varepsilon u \partial_x J_\varepsilon h + \partial_x J_\varepsilon u J_\varepsilon h] - (1 - \partial_x^2)^{-1} \partial_x [2uh + \partial_x u \partial_x h]
$$

$$
\partial_\rho G_\varepsilon(u, \rho) h = -(1 - \partial_x^2) \partial_x [\sigma \rho h]
$$

$$
\partial_u E_\varepsilon(u, \rho) h = -J_\varepsilon [\partial_x J_\varepsilon \rho J_\varepsilon h] - J_\varepsilon [J_\varepsilon \rho \partial_x J_\varepsilon h]
$$

$$
\partial_\rho E_\varepsilon(u, \rho) h = -J_\varepsilon [J_\varepsilon \rho \partial_x J_\varepsilon h] - J_\varepsilon [\partial_x J_\varepsilon \rho J_\varepsilon h].
$$

So we can conclude that $G_\varepsilon$ and $E_\varepsilon$ are continuously differentiable functions with partial derivatives at $u \in H^s$ and $\rho \in H^{s-1}$. Therefore, (3.14) defines a system of ODEs on $H^s \times H^{s-1}$ and thus has a unique solution $(u_\varepsilon, \rho_\varepsilon)$ with lifespan $T_\varepsilon > 0$.

**Energy Estimate** For our energy estimate, we have the following lemma.

**Lemma 5.**

$$
\frac{1}{2} \frac{d}{dt} \| u_\varepsilon \|_{H^s}^2 \leq c_s \left( \| u_\varepsilon \|_{H^s}^3 + \| u_\varepsilon \|_{H^s} \| \rho_\varepsilon \|_{H^{s-1}}^2 \right) \tag{3.16}
$$

$$
\frac{1}{2} \frac{d}{dt} \| \rho_\varepsilon \|_{H^{s-1}}^2 \leq 2c_s \| u_\varepsilon \|_{H^s} \| \rho_\varepsilon \|_{H^{s-1}}^2 \tag{3.17}
$$

**Proof.** Applying the operator $D^*$ to both sides of the first equation in (3.14), multi-
plying the resulting equation by $D^s u_\varepsilon$ and then integrating over the torus yields the following $H^s$ energy of $u_\varepsilon$ identity

$$\frac{1}{2} \frac{d}{dt} \|u_\varepsilon(t)\|_{H^s}^2 = - \int_T D^s J_\varepsilon [J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon] D^s u_\varepsilon dx - \int_T D^s F(u_\varepsilon, \rho_\varepsilon) D^s u_\varepsilon dx$$

(3.18)

To bound (3.18) we need the following commutator estimate.

Lemma 6. (Kato-Ponce) If $s > 0$ then there is a $c_s > 0$ such that

$$\| [D^s, f] g \|_{L^2} \leq c_s (\| D^s f \|_{L^2} \| g \|_{L^\infty} + \| \partial_x f \|_{L^\infty} \| D^{s-1} g \|_{L^2}).$$

We now rewrite the first term of (3.18) by first commuting the exterior $J_\varepsilon$ and then commuting the operator $D^s$ with a $J_\varepsilon u_\varepsilon$ arriving at

$$\int_T D^s [J_\varepsilon u_\varepsilon \partial_x J_\varepsilon u_\varepsilon] D^s J_\varepsilon u_\varepsilon dx = \int_T [D^s, J_\varepsilon u_\varepsilon] \partial_x J_\varepsilon u_\varepsilon D^s J_\varepsilon u_\varepsilon dx + \int_T J_\varepsilon u_\varepsilon \partial_x D^s J_\varepsilon u_\varepsilon D^s J_\varepsilon u_\varepsilon dx$$

(3.19)

Setting $v = J_\varepsilon u_\varepsilon$, we can bound the first term of (3.19) by first using the Cauchy-Schwarz inequality and then applying Lemma 6 to obtain

$$\int_T [D^s, v] \partial_x v D^s v dx \leq \| [D^s, v] \partial_x v \|_{L^2} \| D^s v \|_{L^2}$$

$$\lesssim (\| D^s v \|_{L^2} \| \partial_x v \|_{L^\infty} + \| \partial_x v \|_{L^\infty} \| D^{s-1} \partial_x v \|_{L^2}) \| v \|_{H^s}$$

$$\lesssim \| v \|_{H^s}^2 \| v \|_{H^s}$$

$$= \| v \|_{H^s}^3.$$

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For the second term in (3.19) we have

\[
\int_\mathbb{T} v \partial_x D^s v D^s v dx = - \int_\mathbb{T} \partial_x v (D^s v)^2 dx \\
\leq \sup |\partial_x v| \int_\mathbb{T} (D^s v)^2 dx \\
\leq \|v\|_{C^1} \|v\|_{H^s}^2 \\
\lesssim \|v\|_{H^s}^3.
\]

For the second term of (3.18) we use the Cauchy-Schwarz inequality and Lemma 1 to obtain

\[
\int_\mathbb{T} D^s F(u_\varepsilon, \rho_\varepsilon) D^s u_\varepsilon dx \leq \|D^s F(u_\varepsilon, \rho_\varepsilon)\|_{L^2} \|D^s u_\varepsilon\|_{L^2} \\
\leq \|u_\varepsilon\|_{H^s} \|F(u_\varepsilon, \rho_\varepsilon)\|_{H^s} \\
\lesssim \|u_\varepsilon\|_{H^s} \left(\|u_\varepsilon\|_{H^s}^2 + \|\rho_\varepsilon\|_{H^{s-1}}^2\right) \\
\lesssim \|u_\varepsilon\|_{H^s}^3 + \|u_\varepsilon\|_{H^s} \|\rho_\varepsilon\|_{H^{s-1}}^2
\]

Combining these results, we obtain (3.16) as stated in Lemma 4.

Now, applying $D^{s-1}$, multiplying by $D^{s-1} \rho_\varepsilon$ and integrating over the torus on the second equation of (3.14) gives us

\[
\frac{1}{2} \frac{d}{dt} \|\rho_\varepsilon\|_{H^{s-1}}^2 = - \int_\mathbb{T} D^{s-1} J_\varepsilon \partial_x [J_\varepsilon u_\varepsilon J_\varepsilon \rho_\varepsilon] D^{s-1} \rho_\varepsilon dx. \tag{3.20}
\]
Setting $v = J_\varepsilon u_\varepsilon$ and $w = J_\varepsilon \rho_\varepsilon$ we have that (3.20) becomes

$$\frac{1}{2} \frac{d}{dt} \|\rho_\varepsilon\|^2_{H^{s-1}} = - \int_T D^{s-1} J_\varepsilon \partial_x [vw] D^{s-1} \rho_\varepsilon dx.$$

(3.21)

Now we wish to utilize the following Calderon-Coifman-Meyer type commutator estimate as found in Taylor [52].

**Lemma 7.** If $\gamma + 1 \geq 0$ then

$$\| [D^\gamma \partial_x, f] v \|_{L^2} \leq C \|f\|_{H^k} \|v\|_{H^{\gamma}}$$

provided that $k > 3/2$ and $\gamma + 1 \leq k$.

To bound (3.21) we first commute $J_\varepsilon$ to our $\rho_\varepsilon$. Then we commute $D^{s-1} \partial_x$ with $v$ and obtain the following.

$$\int_T D^{s-1} J_\varepsilon \partial_x [vw] D^{s-1} w dx = \int_T [D^{s-1} \partial_x, v] w D^{s-1} w dx + \int_T v D^{s-1} \partial_x w D^{s-1} w dx.$$

(3.22)

For the first integral on the right hand side in (3.22) we apply Cauchy-Schwarz and Lemma 6 to obtain

$$\int_T [D^{s-1} \partial_x, v] w D^{s-1} w dx \leq \|[D^{s-1} \partial_x, v] w\|_{L^2} \|D^{s-1} w\|_{L^2} \lesssim \|v\|_{H^s} \|w\|^2_{H^{s-1}}.$$
and Cauchy-Schwarz to obtain

\[
\int_T v D^{s-1} \partial_x w D^{s-1} w dx = - \int_T \partial_x v (D^{s-1} w)^2 dx \\
\lesssim \|\partial_x v\|_{L^\infty} \|w\|_{H^{s-1}}^2 \\
\lesssim \|v\|_{H^s} \|w\|_{H^{s-1}}^2.
\]

Combining the above results, we acquire (3.17) as stated in Lemma 4. This concludes the proof.

Now let \(x(t) = \|u_\varepsilon\|_{H^s}\) and \(y(t) = \|\rho_\varepsilon\|_{H^{s-1}}\) to obtain the following system of nonlinear ODEs from Lemma 3

\[
\frac{1}{2} \frac{d}{dt} x^2 \leq c \left( x^3 + xy^2 \right) \tag{3.23}
\]
\[
\frac{1}{2} \frac{d}{dt} y^2 \leq 2c \left( xy^2 \right). \tag{3.24}
\]

If we carry out the differentiation on the left hand side of (3.23) and (3.24) we have

\[
x \frac{dx}{dt} \leq c \left( x^3 + xy^2 \right) \tag{3.25}
\]
\[
y \frac{dy}{dt} \leq 2c \left( xy^2 \right). \tag{3.26}
\]

Dividing out by \(x(t)\) in (3.25), \(y(t)\) in (3.26), assuming that both \(x(t)\) and \(y(t)\) are not zero, and adding both (3.25) and (3.26) together we obtain

\[
\frac{d}{dt} (x + y) \leq c(x + y)^2.
\]
Now let $z = x + y$. Then we have the following ODE:

\[
\frac{dz}{dt} \leq c \cdot z^2.
\]

Solving the above ODE, we obtain

\[
\frac{dz}{dt} \leq c \cdot z^2 \implies z^{-2} \frac{dz}{dt} \leq c \implies \int_0^t z^{-2} \frac{dz}{dt} \, dt \leq \int_0^t c \, dt
\]

\[
\frac{1}{z(0)} - \frac{1}{z(t)} \leq c \cdot t \implies z(t) \leq \frac{z(0)}{1 - c \cdot z(0) \cdot t}.
\]

Thus we have

\[
\|(u_\varepsilon(t), \rho_\varepsilon(t))\|_s \leq \frac{\|(u_0, \rho_0)\|_s}{1 - c \cdot \|(u_0, \rho_0)\|_s} \cdot t.
\]

This implies that solutions $(u_\varepsilon(t), \rho_\varepsilon(t))$ exist till time

\[
T_0 = \frac{1}{c \cdot (\|(u_0, \rho_0)\|_s)}.
\]

Now, if we take $T = \frac{1}{2} T_0$, then we have that solutions $(u_\varepsilon, \rho_\varepsilon)$ exist for $0 \leq t \leq T$ and satisfies a solution size bound

\[
\|(u_\varepsilon(t), \rho_\varepsilon(t))\|_s \leq 2 \|(u_0, \rho_0)\|_s.
\]

Furthermore, we have that

\[
\|\partial_t u_\varepsilon\|_{H^{s-1}} \lesssim \|u_\varepsilon\|_{H^s}^2 + \|\rho_\varepsilon\|_{H^{s-1}}^2 \tag{3.27}
\]

\[
\|\partial_t \rho_\varepsilon\|_{H^{s-2}} \lesssim \|u_\varepsilon\|_{H^s} \|\rho_\varepsilon\|_{H^{s-1}}. \tag{3.28}
\]
Adding \((3.27)\) and \((3.28)\) together we have

\[
\| (\partial_t u_\varepsilon, \partial_t \rho_\varepsilon) \|_{s-1} \lesssim \| (u_\varepsilon(t), \rho_\varepsilon(t)) \|^2_s \lesssim \| (u_0, \rho_0) \|^2_s.
\]
3.2 EXISTENCE

**Proposition 1. (Existence)** There exists a solution \((u,\rho) \in C([0,T];H^s \times H^{s-1})\) to the Cauchy problem for the 2-component CH system \((2.1)\) satisfying the solution size estimate given in \((2.3)\).

**Proof.** We begin by defining the interval \(I = [0,T]\) in order to simplify notation. Our proof revolves around refining the convergence of the family \(\{(u_\epsilon,\rho_\epsilon)\}\) several times by extracting subsequences \(\{(u_{\epsilon,\nu},\rho_{\epsilon,\nu})\}\). After each such extraction, it is assumed that the resulting subsequence is relabeled \(\{(u_\epsilon,\rho_\epsilon)\}\).

**Weak* convergence in** \(L^\infty(I;H^s \times H^{s-1})\). The family \(\{(u_\epsilon,\rho_\epsilon)\}\) is bounded in the space \(C(I;H^s \times H^{s-1}) \subset L^\infty(I;H^s \times H^{s-1})\). If we observe that the dual of the space \(L^1(I,H^s \times H^{s-1})\) is \(L^\infty(I,H^s \times H^{s-1})\), then Alaoglu’s theorem tells us that \(\{(u_\epsilon,\rho_\epsilon)\}\) will be precompact in \(\bar{B}(0,2(\|u_0\|_{H^s(T)}+\|\rho_0\|_{H^{s-1}(T)})) \subset L^\infty(I,H^s \times H^{s-1})\) with respect to the weak* topology. Therefore, we may find a subsequence \(\{(u_{\epsilon,\nu},\rho_{\epsilon,\nu})\}\) that converges to an element in \(\bar{B}(0,2(\|u_0\|_{H^s}+\|\rho_0\|_{H^{s-1}}))\) weakly*.

**Note:** In our sense, we let \((\varphi,\psi) \in L^1(I,H^s \times H^{s-1})\). Then we define the duality in the following manner:

\[
<(u_\epsilon,\rho_\epsilon),(\varphi,\psi)> = \int_0^T <u_\epsilon,\varphi>_H^s \, dt + \int_0^T <\rho_\epsilon,\psi>_H^{s-1} \, dt.
\]

Then we say that we have a subsequence converging weakly* in the following sense:

\[
<(u_{\epsilon,\nu},\rho_{\epsilon,\nu}),(\varphi,\psi) > \rightarrow <(u,\rho),(\varphi,\psi)>.
\]
Relabeling \( \{(u_{\varepsilon}, \rho_{\varepsilon})\} \) as \( \{(u_{\varepsilon}, \rho_{\varepsilon})\} \) we will refine this sequence to converge strongly to \((u, \rho)\) in the \( C(I; H^{s-1} \times H^{s-2}) \) topology. For this we will need the following analysis result:

**Theorem 5.** (Ascoli) Let \( X \) be a Banach space, \( M \) be a compact metric space, and \( C(M, X) \) be the set of continuous functions \( f : M \to X \). If \( S \subset C(M, X) \) is such that

1. \( S \) is equicontinuous.

2. For each \( x \in M \) the set \( S(x) = \{ f(x) \} \) is precompact in \( X \).

Then \( S \) is precompact in \( C(M, X) \).

**Lemma 8.** There exists a subsequence \( \{(u_{\varepsilon}, \rho_{\varepsilon})\} \) of \( \{(u_{\varepsilon}, \rho_{\varepsilon})\} \) that converges strongly to \((u, \rho)\) in the space \( C(I; H^{s-1} \times H^{s-2}) \).

**Proof.** We will prove that the family \( \{(u_{\varepsilon}, \rho_{\varepsilon})\} \in [0,1] \) satisfies the hypothesis of Ascoli’s Theorem. We begin with the equicontinuity condition. For \( t_1, t_2 \in I \) we have by the Mean Value Theorem

\[
\| u_{\varepsilon}(t_1) - u_{\varepsilon}(t_2) \|_{H^{s-1}} \leq \sup_{t \in I} \| \partial_t u_{\varepsilon}(t) \|_{H^{s-1}} |t_1 - t_2| 
\]

\[
\| \rho_{\varepsilon}(t_1) - \rho_{\varepsilon}(t_2) \|_{H^{s-2}} \leq \sup_{t \in I} \| \partial_t \rho_{\varepsilon}(t) \|_{H^{s-2}} |t_1 - t_2| .
\]

Now using the bounds (3.27) and (3.28) for \( \| \partial_t u_{\varepsilon}(t) \|_{H^{s-1}} \) and \( \| \partial_t \rho_{\varepsilon}(t) \|_{H^{s-2}} \) we obtain
the inequality

\[ \|(u_\varepsilon(t_1) - u_\varepsilon(t_2), \rho_\varepsilon(t_1) - \rho_\varepsilon(t_2))\|_{s-1} \lesssim |t_1 - t_2|, \]

which establishes equicontinuity. We have that for each \( t \in I \), the set \( U(t) = \{(u_\varepsilon(t), \rho_\varepsilon(t))\}_{\varepsilon \in (0,1]} \) is bounded in \( H^s \times H^{s-1} \). Since \( \mathbb{T} \) is a compact manifold, the inclusion mapping \( i : H^s \times H^{s-1} \to H^{s-1} \times H^{s-2} \) is a compact operator, and therefore we may deduce that \( U(t) \) is a precompact set in \( H^{s-1} \times H^{s-2} \). Thus there is a subsequence of \( \{(u_\varepsilon, \rho_\varepsilon)\} \) converging to an element in \( H^{s-1} \times H^{s-2} \). By the uniqueness of the limit, this element is equal to \((u, \rho)\).

Our next goal is to demonstrate that the sequence \((u_\varepsilon, \rho_\varepsilon) \to (u, \rho)\) converges strongly in the space \( C(I; H^{s-\gamma} \times H^{s-\gamma-1}) \) for \( \gamma \in (0,1) \). To accomplish this task we first need an interpolation result.

**Lemma 9.** For \( \varepsilon \in (0,1] \) and \( \gamma \in (0,1) \) we have \((u_\varepsilon, \rho_\varepsilon) \in C^\gamma(I; H^{s-\gamma} \times H^{s-\gamma-1})\). Furthermore, the \( C^\gamma(I; H^{s-\gamma} \times H^{s-\gamma-1}) \) norm of \((u_\varepsilon, \rho_\varepsilon)\) is bounded by

\[
\|u_\varepsilon\|_{C^\gamma(I; H^{s-\gamma})} + \|\rho_\varepsilon\|_{C^\gamma(I; H^{s-\gamma-1})} \lesssim \|(u_0, \rho_0)\|_s + \|(u_0, \rho_0)\|_s^2 \tag{3.31}
\]

**Proof.** By definition we have

\[
\|u_\varepsilon\|_{C^\gamma(I; H^{s-\gamma})} \doteq \sup_{t \in I} \|u_\varepsilon(t)\|_{H^{s-\gamma}} + \sup_{t \neq t'} \frac{\|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{s-\gamma}}}{|t - t'|^\gamma} \tag{3.32}
\]

\[
\|\rho_\varepsilon\|_{C^\gamma(I; H^{s-\gamma-1})} \doteq \sup_{t \in I} \|\rho_\varepsilon(t)\|_{H^{s-\gamma-1}} + \sup_{t \neq t'} \frac{\|\rho_\varepsilon(t) - \rho_\varepsilon(t')\|_{H^{s-\gamma-1}}}{|t - t'|^\gamma}. \tag{3.33}
\]
First, we will note that

\[
\sup_{t \in I} \| u_\varepsilon \|_{H^{s-\gamma}} \leq \| u_\varepsilon \|_{H^s} \quad \text{and} \quad \sup_{t \in I} \| \rho_\varepsilon \|_{H^{s-\gamma-1}} \leq \| \rho_\varepsilon \|_{H^{s-1}}.
\]

Using our bound from (2.3) we have

\[
\sup_{t \in I} \| u_\varepsilon \|_{H^{s-\gamma}} + \sup_{t \in I} \| \rho_\varepsilon \|_{H^{s-\gamma-1}} \leq 2 \left( \| u_0 \|_{H^s} + \| \rho_0 \|_{H^{s-1}} \right).
\]

(3.34)

For the right term of (3.32), using the inequality \( x^\gamma \leq 1 + x, \ x > 0 \), we have

\[
\sup_{t \neq t'} \frac{\| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}}^2}{|t - t'|^{2\gamma}} = \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-\gamma} \frac{|\hat{u}_\varepsilon(t) - \hat{u}_\varepsilon(t')|^2}{|t - t'|^{2\gamma}} \leq \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-\gamma} |\hat{u}_\varepsilon(t) - \hat{u}_\varepsilon(t')|^2 + \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-1} \frac{|\hat{u}_\varepsilon(t) - \hat{u}_\varepsilon(t')|^2}{|t - t'|^{2}} \lesssim \| u_\varepsilon \|_{C(I; H^s)}^2 + \| \partial_t u_\varepsilon \|_{C(I; H^{s-1})}^2 \lesssim \| u_\varepsilon \|_{H^s}^2 + \left( \| u_\varepsilon \|_{H^s}^2 + \| \rho_\varepsilon \|_{H^{s-1}}^2 \right)^2.
\]

(3.35)

Taking the square root of both sides of (3.35) we obtain

\[
\sup_{t \neq t'} \frac{\| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}}}{|t - t'|^{\gamma/2}} \lesssim \| u_\varepsilon \|_{H^s} + \left( \| u_\varepsilon \|_{H^s}^2 + \| \rho_\varepsilon \|_{H^{s-1}}^2 \right)^{1/2}.
\]

(3.36)

Using the same analysis above, we obtain the following inequality for the right side
of (3.33)
\[ \sup_{t \neq t'} \frac{\| \rho_\epsilon(t) - \rho_\epsilon(t') \|_{H^{s-\gamma-1}}^2}{|t - t'|^{2\gamma}} = \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-\gamma-1} \frac{|\tilde{\rho}_\epsilon(t) - \tilde{\rho}_\epsilon(t')|^2}{|t - t'|^{2\gamma}} \leq \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-\gamma-1} |\tilde{\rho}_\epsilon(t) - \tilde{\rho}_\epsilon(t')|^2 
+ \sup_{t \neq t'} \sum_{k \in \mathbb{Z}} \frac{(1 + k^2)^{s-\gamma-2}}{|t - t'|^2} |\tilde{\rho}_\epsilon(t) - \tilde{\rho}_\epsilon(t')|^2 \lesssim \| \rho_\epsilon \|^2_{C(I; H^{s-1})} + \| \partial_t \rho_\epsilon \|^2_{C(I; H^{s-2})} \lesssim \| \rho_\epsilon \|^2_{H^{s-1}} + \| u_\epsilon \|^2_{H^s} \| \rho_\epsilon \|^2_{H^{s-1}} \] (3.37)

Taking the square root of both sides of (3.37) we obtain
\[ \sup_{t \neq t'} \frac{\| \rho_\epsilon(t) - \rho_\epsilon(t') \|_{H^{s-\gamma-1}}}{|t - t'|^{\gamma}} \lesssim \| \rho_\epsilon \|_{H^{s-1}} + \| u_\epsilon \|^2_{H^s} \| \rho_\epsilon \|^2_{H^{s-1}}. \] (3.38)

Combining (3.34), (3.36) and (3.38) gives us our desired bound. \( \square \)

**Lemma 10.** For \( \gamma \in (0,1) \), there exists a subsequence \((u_{\epsilon \nu}, \rho_{\epsilon \nu})\) of \((u_\epsilon, \rho_\epsilon)\) that converges strongly to \((u, \rho)\) in the \(C(I; H^{s-\gamma} \times H^{s-\gamma-1})\) topology. Furthermore, we have that \((u_{\epsilon \nu}, \rho_{\epsilon \nu}) \to (u, \rho)\) in \(C(I, C^1 \times C^1)\).

**Proof.** We will once again use Ascoli’s Theorem. The precompactness condition is established with the same argument as Lemma 8. For the condition of equicontinuity
we observe have

\[\|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{s-\gamma}} \leq \|u_\varepsilon\|_{C^\gamma(I; H^{s-\gamma})}|t - t'|^\gamma \] (3.39)

\[\|\rho_\varepsilon(t) - \rho_\varepsilon(t')\|_{H^{s-\gamma-1}} \leq \|\rho_\varepsilon\|_{C^\gamma(I; H^{s-\gamma-1})}|t - t'|^\gamma. \] (3.40)

Adding (3.39) and (3.40) together and using the bound (3.31) we find that

\[\|(u_\varepsilon(t) - u_\varepsilon(t'), \rho_\varepsilon(t) - \rho_\varepsilon(t'))\|_{s-\gamma} \lesssim |t - t'|^\gamma, \] (3.41)

which establishes equicontinuity. We have that for each \(t \in I\), the set \(U(t) = \{(u_\varepsilon(t), \rho_\varepsilon(t))\}_{\varepsilon \in (0,1]}\) is bounded in \(H^{s-\gamma} \times H^{s-\gamma-1}\). Since \(\mathbb{T}\) is a compact manifold, the inclusion mapping \(i : H^s \times H^{s-1} \to H^{s-\gamma} \times H^{s-\gamma-1}\) is a compact operator, and therefore we have that \(U(t)\) is a precompact set in \(H^{s-\gamma} \times H^{s-\gamma-1}\). Thus there is a subsequence of \(\{(u_{\varepsilon_\nu}, \rho_{\varepsilon_\nu})\}\) converging to an element in \(H^{s-\gamma} \times H^{s-\gamma-1}\). By the uniqueness of the limit, this element is equal to \((u, \rho)\). Therefore, \((u_{\varepsilon_\nu}, \rho_{\varepsilon_\nu})\) of \((u_\varepsilon, \rho_\varepsilon)\) that converges strongly to \((u, \rho)\) in the \(C(I; H^{s-\gamma} \times H^{s-\gamma-1})\) topology.

**Equicontinuity of \(\{(u_\varepsilon, \rho_\varepsilon)\}\) in \(C(I, C^1 \times C^1)\).** Choosing \(\gamma > 0\) small enough so that \(s - \gamma > 5/2\) we can apply the Sobolev Embedding Theorem to obtain

\[\|u_\varepsilon(t) - u_\varepsilon(t')\|_{C^1} \lesssim \|u_\varepsilon(t) - u_\varepsilon(t')\|_{H^{s-\gamma}} \] (3.42)

\[\|\rho_\varepsilon(t) - \rho_\varepsilon(t')\|_{C^1} \lesssim \|\rho_\varepsilon(t) - \rho_\varepsilon(t')\|_{H^{s-\gamma-1}}. \] (3.43)
Adding (3.42) and (3.43) and using (3.41) we have
\[ \|u_\varepsilon(t) - u_\varepsilon(t')\|_{C^1} + \|\rho_\varepsilon(t) - \rho_\varepsilon(t')\|_{C^1} \lesssim |t - t'|^\gamma, \]
which shows that the family \{\(u_\varepsilon, \rho_\varepsilon\)\} is equicontinuous is \(C(I, C^1 \times C^1)\).

**Compactness of \{\(u_\varepsilon, \rho_\varepsilon\)\} in \(C^1 \times C^1\).** Using estimate (2.3) gives
\[ \|(u_\varepsilon(t), \rho_\varepsilon(t))\|_{s-\gamma} \leq \|(u_\varepsilon(t), \rho_\varepsilon(t))\|_s \leq 2\|(u_0, \rho_0)\|_s, \]
which is finite. Therefore, by Rellich’s Lemma for each \(t \in I\) the set \{\(u_\varepsilon, \rho_\varepsilon\)\} is precompact in \(H^{s-\gamma} \times H^{s-\gamma-1}\). That is, any subset of \{\(u_\varepsilon, \rho_\varepsilon\)\} contains a sequence \{\(u_{\varepsilon_n}, \rho_{\varepsilon_n}\)\} which converges to an element \((u, \rho) \in H^{s-\gamma} \times H^{s-\gamma-1}\). Choosing \(\gamma > 0\) small enough so that \(s - \gamma > 5/2\) and applying the Sobolev Embedding Theorem gives
\[ \|u_{\varepsilon_n}(t) - u_\varepsilon(t)\|_{C^1} + \|\rho_{\varepsilon_n}(t) - \rho_\varepsilon(t)\|_{C^1} \to 0 \text{ as } n \to \infty, \]
which shows that for each \(t \in I\) the family \{\(u_\varepsilon, \rho_\varepsilon\)\} is precompact in \(C^1 \times C^1\). Therefore, \((u_{\varepsilon_n}, \rho_{\varepsilon_n}) \to (u, \rho)\) in \(C(I, C^1 \times C^1)\).

**Verifying that the limit \((u, \rho)\) solves the 2-component CH system.** Before we begin, we will need the following Lemma and Theorem.

**Lemma 11.** For \(0 < r \leq s\), the map \(I - J_\varepsilon : H^s \to H^r\) satisfies the norm operator estimate
\[ \|I - J_\varepsilon\|_{\mathcal{L}(H^s, H^r)} = o(\varepsilon^{s-r}). \]

**Proof.** Let \(f \in H^s\), then applying the \(H^r\) norm to \((I - J_\varepsilon)f\) and using the
Dominated Convergence Theorem we have

\[
\lim_{\varepsilon \to 0} \frac{\| f - J_{\varepsilon} f \|_{H^r}}{\varepsilon^{2(s-r)}} = \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}} \left| \frac{\hat{f}(\varepsilon k) - 1}{\varepsilon^{2(s-r)}} \right| \left| \frac{\hat{f}(k)}{1 + k^2} \right|^2 (1 + k^2)^s
\]

\[
\leq \lim_{\varepsilon \to 0} \sum_{k \in \mathbb{Z}, |k| > 1} \left| \frac{\hat{f}(\varepsilon k) - 1}{\varepsilon^{2(s-r)}} \right| \left| \hat{f}(k) \right|^2 (1 + k^2)^s = 0. \tag*{\Box}
\]

**Theorem 6.** [Sequences of differentiable functions] Let \( A \) be an open and connected subset of a Banach space \( X \) and \( f_n : A \to Y \) be a sequence of differentiable functions from \( A \) into a Banach space \( Y \). Suppose that:

1. there is a \( x_0 \in A \) such that the sequence \( \{ f_n(x_0) \} \) converges in \( Y \);
2. For every \( a \in A \) there is a ball \( B(a) \) centered at \( a \) and contained in \( A \) such that \( \{ f'_n \} \) converges uniformly in \( B(a) \).

Then for each \( a \in A \) the sequence \( \{ f_n \} \) converges uniformly in \( B(a) \). Moreover, if for each \( x \in A \), \( f(x) = \lim_{n \to \infty} f_n(x) \) and \( g(x) = \lim_{n \to \infty} f'_n(x) \), then \( g(x) = f'(x) \) for every \( x \in A \).

Our next lemma examines the convergence of \( \partial_t u_{\varepsilon_v} \) and \( \partial_t \rho_{\varepsilon_v} \).

**Lemma 12.** The sequence \( \{ (\partial_t u_{\varepsilon_v}, \partial_t \rho_{\varepsilon_v}) \} \to (\partial_t u, \partial_t \rho) \) in \( C(I, C \times C) \).
Starting from the mollified i.v.p. (3.14) we have

\[
\partial_t u_{\varepsilon} = -J_{\varepsilon}[J_{\varepsilon} u_{\varepsilon} \partial_x J_{\varepsilon} u_{\varepsilon}] - D^{-2}\partial_x (u_{\varepsilon}^2) - \frac{1}{2} D^{-2}\partial_x [(\partial_x u_{\varepsilon})^2] - \frac{\sigma}{2} D^{-2}\partial_x (\rho_{\varepsilon}^2)
\]

(3.44)

\[
\partial_t \rho_{\varepsilon} = -J_{\varepsilon}[J_{\varepsilon} u_{\varepsilon} \partial_x J_{\varepsilon} \rho_{\varepsilon}] - J_{\varepsilon}[J_{\varepsilon} \rho_{\varepsilon} \partial_x J_{\varepsilon} u_{\varepsilon}].
\]

(3.45)

By the continuity of the operator \(D^{-2}\partial_x\), we may immediately deduce the convergence of the nonlocal terms of (3.44) as

\[
D^{-2}\partial_x (u_{\varepsilon}^2) \rightarrow D^{-2}\partial_x (u^2)
\]

\[
\frac{1}{2} D^{-2}\partial_x [(\partial_x u_{\varepsilon})^2] \rightarrow \frac{1}{2} D^{-2}\partial_x [(\partial_x u)^2]
\]

\[
\frac{\sigma}{2} D^{-2}\partial_x (\rho_{\varepsilon}^2) \rightarrow \frac{\sigma}{2} D^{-2}\partial_x (\rho^2).
\]

To handle the mollified Burgers term of (3.44), we will first prove that \(J_{\varepsilon} u_{\varepsilon} \rightarrow u\) in \(C(I, C)\). We have

\[
\|J_{\varepsilon} u_{\varepsilon} - u\|_{C(I, C)} \leq \|J_{\varepsilon} u_{\varepsilon} - u_{\varepsilon} \|_{C(I, C)} + \|u_{\varepsilon} - u\|_{C(I, C)}.
\]

(3.46)

For the first term of (3.46), choose \(1/2 < r < s\). Then by Lemma [11] for \(t \in I\) we
have

\[ \| J_{\varepsilon} u_{\varepsilon} - u_{\varepsilon} \|_{C(I,C)} \lesssim \| J_{\varepsilon} u_{\varepsilon} - u_{\varepsilon} \|_{H^s} \]

\[ \lesssim \| I - J_{\varepsilon} \|_{L(H^s, H^r)} \| u_{\varepsilon} \|_{H^s} \]

\[ = o(\varepsilon^{s-r}). \]

For the second term of (3.46), we observe that Lemma 10 and the Sobolev Embedding Theorem imply that \( \| u_{\varepsilon} - u \|_{C(I,C)} \to 0 \) and that \( u \in C(I, C) \). Now we examine \( \partial_x u \) as above to obtain

\[ \| J_{\varepsilon} \partial_x u_{\varepsilon} - \partial_x u \|_{C(I,C)} \leq \| J_{\varepsilon} \partial_x u_{\varepsilon} - \partial_x u_{\varepsilon} \|_{C(I,C)} + \| \partial_x u_{\varepsilon} - \partial_x u \|_{C(I,C)}. \quad (3.47) \]

For the first term of (3.47), choose \( 1/2 < r < s - 1 \). Then by Lemma 10, for \( t \in I \) we have

\[ \| J_{\varepsilon} \partial_x u_{\varepsilon} - \partial_x u_{\varepsilon} \|_{C(I,C)} \lesssim \| J_{\varepsilon} \partial_x u_{\varepsilon} - \partial_x u_{\varepsilon} \|_{H^r} \]

\[ \lesssim \| I - J_{\varepsilon} \|_{L(H^{s-1}, H^r)} \| \partial_x u_{\varepsilon} \|_{H^{s-1}} \quad (3.48) \]

\[ = o(\varepsilon^{s-r-1}). \]

For the second term of (3.47), we observe that \( \| u_{\varepsilon} - u \|_{C(I; C^1)} \to 0 \) implies \( \| \partial_x u_{\varepsilon} - \partial_x u \|_{C(I,C)} \to 0 \). Similarly, this can be shown for the mollified terms of (3.45).

Thus, proceeding via additive and multiplicative properties of limits we may conclude that \( \{ (\partial_t u_{\varepsilon}, \partial_t \rho_{\varepsilon}) \} \to (\partial_t u, \partial_t \rho) \) in \( C(I, C \times C) \). \( \Box \)

**Improving the regularity of \((u, \rho)\) to \(C(I; H^s \times H^{s-1})\).** Now that we have
established the existence of a solution \((u, \rho) \in L^\infty(I; H^s \times H^{s-1})\) our next step is to improve its regularity by showing that it belongs to the space \(C(I; H^s \times H^{s-1})\).

**Proposition 2.** The solution \((u, \rho)\) to the 2-component CH system i.v.p. is an element of the space \(C(I; H^s \times H^{s-1})\).

**Proof.** We must establish the fact that \((u, \rho) \in L^\infty(I; H^s \times H^{s-1})\) is in fact continuous. Fix \(t \in I\) and take a sequence \(\{t_n\} \to t\). For the solution \((u, \rho)\) to be continuous at \(t\), we must have \(\|u(t_n) - u(t)\|_{H^s} \to 0\) and \(\|\rho(t_n) - \rho(t)\|_{H^{s-1}} \to 0\). Next we observe that

\[
\|u(t_n) - u(t)\|_{H^s}^2 + \|\rho(t_n) - \rho(t)\|_{H^{s-1}}^2 = \|u(t_n)\|_{H^s}^2 + \|u(t)\|_{H^s}^2 \\
- \langle u(t_n), u(t) \rangle_{H^s} \\
- \langle u(t), u(t_n) \rangle_{H^s} \\
+ \|\rho(t_n)\|_{H^{s-1}}^2 + \|\rho(t)\|_{H^{s-1}}^2 \\
- \langle \rho(t_n), \rho(t) \rangle_{H^{s-1}} \\
- \langle \rho(t), \rho(t_n) \rangle_{H^{s-1}}
\]

Therefore, to complete the proof of Proposition 2 it suffices to prove the following two lemmas.

**Lemma 13.** The solution \((u, \rho) \in L^\infty(I; H^s \times H^{s-1}) \cap \text{Lip}(I; H^{s-1} \times H^{s-2})\) is continuous in \(t\) with respect to the weak topology on \(H^s \times H^{s-1}\). That is, for any \(\phi \in H^s\)
and \( \gamma \in H^{s-1} \) and a sequence \( t_n \to t \) we have that
\[
\langle u(t_n) - u(t), \phi \rangle_{H^s} + \langle \rho(t_n) - \rho(t), \gamma \rangle_{H^{s-1}} \to 0.
\]

Applying this lemma with \( \phi = u \) and \( \gamma = \rho \) gives
\[
\lim_{n \to \infty} \langle u(t_n), u(t) \rangle_{H^s} = \| u(t) \|_{H^s}^2
\]
\[
\lim_{n \to \infty} \langle \rho(t_n), \rho(t) \rangle_{H^{s-1}} = \| \rho(t) \|_{H^{s-1}}^2
\]
which are the first desired relations.

**Lemma 14.** For the solution \((u, \rho) \in L^\infty(I; H^s \times H^{s-1}) \cap Lip(I; H^{s-1} \times H^{s-2})\) to the 2-component CH system Cauchy problem we have
\[
\lim_{n \to \infty} \| u(t_n) \|_{H^s}^2 = \| u(t) \|_{H^s}^2
\]
\[
\lim_{n \to \infty} \| \rho(t_n) \|_{H^{s-1}}^2 = \| \rho(t) \|_{H^{s-1}}^2
\]

**Proof of Lemma 13** Let \( \varepsilon > 0 \). We will choose a \( \psi \in \mathcal{S}(\mathbb{R}) \) with \( \| \phi - \psi \|_{H^s}, \| \gamma - \psi \|_{H^{s-1}} < \varepsilon/8 \) \( (\| u_0 \|_{H^s} + \| \rho_0 \|_{H^{s-1}}) \). Then we see that
\[
|\langle u(t_n) - u(t), \phi \rangle_{H^s}| \leq |\langle u(t_n) - u(t), \phi - \psi \rangle_{H^s}| + |\langle u(t_n) - u(t), \psi \rangle_{H^s}| \quad (3.49)
\]
and
\[
|\langle \rho(t_n) - \rho(t), \gamma \rangle_{H^{s-1}}| \leq |\langle \rho(t_n) - \rho(t), \gamma - \psi \rangle_{H^{s-1}}| + |\langle \rho(t_n) - \rho(t), \psi \rangle_{H^{s-1}}|. \quad (3.50)
\]
For the first term of (3.49) we obtain

\[ |\langle u(t_n) - u(t), \phi - \psi \rangle_{H^s} | \leq \|u(t_n) - u(t)\|_{H^s} \|\phi - \psi\|_{H^s} \]
\[ \leq (\|u(t_n)\|_{H^s} + \|u(t)\|_{H^s}) \|\phi - \psi\|_{H^s}. \] (3.51)

Similarly, for the first term of (3.50) we obtain

\[ |\langle \rho(t_n) - \rho(t), \gamma - \psi \rangle_{H^{s-1}} | \leq \|\rho(t_n) - \rho(t)\|_{H^{s-1}} \|\gamma - \psi\|_{H^{s-1}} \]
\[ \leq (\|\rho(t_n)\|_{H^{s-1}} + \|\rho(t)\|_{H^{s-1}}) \|\gamma - \psi\|_{H^{s-1}}. \] (3.52)

For the second term of (3.49) we use the bound from (3.29) to acquire the following bound

\[ |\langle u(t_n) - u(t), \psi \rangle_{H^s} | \leq \|u(t_n) - u(t)\|_{H^{s-1}} \|\psi\|_{H^{s+1}} \]
\[ \leq \sup_{t \in I} \|\partial_t u(t)\|_{H^{s-1}} \|\psi\|_{H^{s+1}} |t_n - t|. \] (3.53)

Similarly for the second term of (3.50) we have

\[ |\langle \rho(t_n) - \rho(t), \psi \rangle_{H^{s-1}} | \leq \|\rho(t_n) - \rho(t)\|_{H^{s-2}} \|\psi\|_{H^s} \]
\[ \lesssim \sup_{t \in I} \|\partial_t \rho(t)\|_{H^{s-2}} \|\psi\|_{H^{s+1}} |t_n - t|. \] (3.54)

Now, by adding (3.51) and (3.52) we find that

\[ |\langle u(t_n) - u(t), \phi - \psi \rangle_{H^s} | + |\langle \rho(t_n) - \rho(t), \gamma - \psi \rangle_{H^{s-1}} | < \frac{\varepsilon}{2}. \]
Also, by adding (3.53) and (3.54) we have

\[ |\langle u(t_n) - u(t), \psi \rangle_{H^s} + |\langle \rho(t_n) - \rho(t), \psi \rangle_{H^{s-1}}| \lesssim \|\psi\|_{H^{s+1}} |t_n - t| \]  \hfill (3.55)

By choosing \( n \) sufficiently large, we can bound (3.55) by \( \varepsilon/2 \). So we achieve the following result

\[ |\langle u(t_n) - u(t), \phi \rangle_{H^s} + |\langle \rho(t_n) - \rho(t), \gamma \rangle_{H^{s-1}}| < \varepsilon. \]

This concludes our proof of Lemma 13. \( \square \)

**Proof of Lemma 14.** Before beginning, we will state the following lemma that we shall use during the proof.

**Lemma 15.** Let \( u(x) \) be a function such that \( \|\partial_x u\|_{L^\infty} < \infty \). Then there is a \( c > 0 \) such that for any \( f \in L^2(\mathbb{R}) \) we have

\[ \| [J_\varepsilon, u] \partial_x f \|_{L^2} \leq c \|\partial_x u\|_{L^\infty} \|f\|_{L^2}. \]  \hfill (3.56)

**Proof of Lemma 15.** We have

\[
[J_\varepsilon, u] \partial_x f(x) = J_\varepsilon (u \partial_x f)(x) - u J_\varepsilon (\partial_x f)(x) \\
= j_\varepsilon * (u \partial_x f)(x) - u(x) j_\varepsilon * (\partial_x f)(x) \\
= \int_T j_\varepsilon (x - y) u(y) f'(y) dy - u(x) \int_T j_\varepsilon (x - y) f'(y) dy \\
= \int_T \frac{1}{\varepsilon^d} \left( \frac{x - y}{\varepsilon} \right) [u(y) - u(x)] f'(y) dy.
\]
Integrating by parts and using the Mean Value Theorem yields

\[
[J_\varepsilon, u] \partial_x f(x) = \int_T^1 \frac{1}{\varepsilon} j \left( \frac{x - y}{\varepsilon} \right) u'(y) f(y) dy \\
- \int_T^1 \frac{1}{\varepsilon^2} j' \left( \frac{x - y}{\varepsilon} \right) [u(y) - u(x)] f(y) dy \\
= \int_{|y-x|<\varepsilon} \frac{1}{\varepsilon} j \left( \frac{x - y}{\varepsilon} \right) u'(y) f(y) dy \\
- \int_{|y-x|<\varepsilon} \frac{1}{\varepsilon^2} j' \left( \frac{x - y}{\varepsilon} \right) u'(\xi(x, y))(y - x) f(y) dy 
\]

Above we have used our assumption that \( j(x) \) is supported on the interval \([-1, 1]\). So, using the bound \(|(x - y)/\varepsilon| < 1\) and taking absolute values we obtain

\[
|[J_\varepsilon, u] \partial_x f(x)| \leq \|\partial_x u\|_{L^\infty} \int_T^1 \frac{1}{\varepsilon} j \left( \frac{x - y}{\varepsilon} \right) |f(y)| dy \\
+ \|\partial_x u\|_{L^\infty} \int_T^1 \frac{1}{\varepsilon} j' \left( \frac{x - y}{\varepsilon} \right) |f(y)| dy \\
= \|\partial_x u\|_{L^\infty} (j_\varepsilon * |f|(x) + j'_\varepsilon * |f|(x)).
\]

Finally, applying Young’s Inequality we have

\[
\| [J_\varepsilon, u] \partial_x f(x) \|_{L^2} \leq \|\partial_x u\|_{L^\infty} (\|j_\varepsilon\|_{L^1} \|f\|_{L^2} + \|j'_\varepsilon\|_{L^1} \|f\|_{L^2}) \\
= (\|j_\varepsilon\|_{L^1} + \|j'_\varepsilon\|_{L^1}) \|\partial_x u\|_{L^\infty} \|f\|_{L^2},
\]

which gives the desired inequality \((3.56)\) with constant \(c = \|j_\varepsilon\|_{L^1} + \|j'_\varepsilon\|_{L^1} \).
Let’s begin by defining the functions

\[ F(t) = \|u(t)\|_{H^s} + \|\rho(t)\|_{H^{s-1}} \]

\[ F_\varepsilon(t) = \|J_\varepsilon u(t)\|_{H^s} + \|J_\varepsilon \rho(t)\|_{H^{s-1}}. \]

Lemma 11 implies that \( F_\varepsilon \to F \) pointwise as \( \varepsilon \to 0 \). We will prove that each \( F_\varepsilon \) is Lipschitz, and that the Lipschitz constants for this family of functions are uniformly bounded by some independent constant. This fact will imply that \( F \) is also Lipschitz continuous and its Lipschitz constant is bounded by this same independent constant which will complete our proof. This will be achieved by energy estimates. Analogous to Himonas and Kenig [Diff. Int. Eq. 2009] we mollify (3.1) by applying to both sides the operator \( J_\varepsilon \). Thus, we obtain

\[ \partial_t J_\varepsilon u = -J_\varepsilon(u \partial_x u) - J_\varepsilon D^{-2} \partial_x \left( u^2 \right) - \frac{1}{2} J_\varepsilon D^{-2} \partial_x \left[ (\partial_x u)^2 \right] - \frac{\sigma}{2} J_\varepsilon D^{-2} \partial_x (\rho^2) \tag{3.57} \]

\[ \partial_t J_\varepsilon \rho = -J_\varepsilon(u \partial_x \rho) - J_\varepsilon(\rho \partial_x u). \tag{3.58} \]

Applying the operator \( D^s \) to both sides of (3.57), multiplying the resulting equation
by $D^s J_\epsilon u$, and integrating over the torus gives

$$
\frac{1}{2} \frac{d}{dt} \| J_\epsilon u \|_{H^s}^2 = - \int_T D^s J_\epsilon (u \partial_x u) D^s J_\epsilon u dx
- \int_T D^{s-2} \partial_x J_\epsilon (u^2) D^s J_\epsilon u dx
- \frac{1}{2} \int_T D^{s-2} \partial_x J_\epsilon [(\partial_x u)^2] D^s J_\epsilon u dx
- \frac{\sigma}{2} \int_T D^{s-2} \partial_x J_\epsilon (\rho^2) D^s J_\epsilon u dx.
$$

(3.59)

In what follows next, we will use the fact that $D^s$ and $J_\epsilon$ commute and that $J_\epsilon$ satisfies

the properties

$$(J_\epsilon f, g)_{L^2} = (f, J_\epsilon g)_{L^2}$$

and

$$\| J_\epsilon u \|_{H^s} \leq \| u \|_{H^s}.$$  

**Estimating the Burgers term.** For the first integral on the right hand side of (3.59) we have

$$
\int_T D^s J_\epsilon (u \partial_x u) D^s J_\epsilon u dx = \int_T D^s (u \partial_x u) J_\epsilon D^s J_\epsilon u dx
= \int_T [D^s, u] \partial_x u J_\epsilon D^s J_\epsilon u dx
+ \int_T u D^s (\partial_x u) J_\epsilon D^s J_\epsilon u dx.
$$

(3.60)
Using the Cauchy-Schwarz inequality and the Kato-Ponce lemma on (3.60) we obtain

\[
\left| \int_T [D^s, u] \partial_x u J_\varepsilon D^s J_\varepsilon u \, dx \right| \leq \| [D^s, u] \partial_x u \|_{L^2} \| J_\varepsilon D^s J_\varepsilon u \|_{L^2} \\
\lesssim \left( \| D^s u \|_{L^2} \| \partial_x u \|_{L^\infty} + \| \partial_x u \|_{L^\infty} \| D^{s-1} \partial_x u \|_{L^2} \right) \| u \|_{H^s} \\
\lesssim \| u \|_{H^s}^2 \| u \|_{H^s} \\
= \| u \|_{H^s}^3. \tag{3.62}
\]

For (3.61) we have

\[
\int_T u D^s (\partial_x u) J_\varepsilon D^s J_\varepsilon u \, dx = \int_T J_\varepsilon u D^s (\partial_x u) D^s J_\varepsilon u \, dx \\
= \int_T (\{ J_\varepsilon, u \} D^s \partial_x u + u J_\varepsilon D^s (\partial_x u)) D^s J_\varepsilon u \, dx \\
= \int_T [J_\varepsilon, u] \partial_x D^s u D^s J_\varepsilon u \, dx \\
+ \int_T u \partial_x D^s J_\varepsilon u D^s J_\varepsilon u \, dx. \tag{3.63}
\]

For the first integral on the right hand side of (3.63) we will use the Cauchy-Schwarz inequality and Lemma 15 to obtain

\[
\left| \int_T [J_\varepsilon, u] \partial_x D^s u D^s J_\varepsilon u \, dx \right| \leq \| [J_\varepsilon, u] \partial_x D^s u \|_{L^2} \| D^s J_\varepsilon u \|_{L^2} \\
\lesssim \| \partial_x u \|_{L^\infty} \| D^s u \|_{L^2} \| u \|_{H^s} \\
\lesssim \| u \|_{H^s}^3. \tag{3.64}
\]

For the second integral in (3.63) we use the Cauchy-Schwarz inequality to yield

\[
\left| \int_T u \partial_x D^s J_\varepsilon u D^s J_\varepsilon u \, dx \right| \leq - \int_T \partial_x u D^s J_\varepsilon u D^s J_\varepsilon u \, dx \leq \| \partial_x u \|_{L^\infty} \| D^s J_\varepsilon u \|_{L^2}^2 \\
\lesssim \| \partial_x u \|_{L^\infty} \| u \|_{H^s}^2 \\
\lesssim \| u \|_{H^s}^3. \tag{3.65}
\]

Combining (3.62), (3.64) and (3.65) we obtain the following estimate for the Burgers term

\[
\left| \int_T D^s J_\varepsilon (u \partial_x u) D^s J_\varepsilon u \, dx \right| \lesssim \| u \|_{H^s}^3. \tag{3.66}
\]

**Estimating the nonlocal** $D^{s-2} \partial_x J_\varepsilon (u^2)$. To estimate the second integral of (3.59) we apply the Cauchy-Schwarz inequality to obtain

\[
\left| \int_T D^{s-2} \partial_x J_\varepsilon (u^2) D^s J_\varepsilon u \, dx \right| \leq \| D^{s-2} \partial_x J_\varepsilon (u^2) \|_{L^2} \| D^s J_\varepsilon u \|_{L^2} \\
\lesssim \| u^2 \|_{H^s} \| u \|_{H^s} \\
= \| u \|_{H^s}^3. \tag{3.67}
\]

**Estimating the nonlocal** $D^{s-2} \partial_x J_\varepsilon [(\partial_x u)^2]$. To estimate the third integral of
we apply the Cauchy-Schwarz inequality to yield
\[
\left| \int_T D^{s-2} \partial_x J_{\varepsilon} \left[ (\partial_x u)^2 \right] D^s J_{\varepsilon} u \, dx \right| \leq \| D^{s-2} \partial_x J_{\varepsilon} \left[ (\partial_x u)^2 \right] \|_{L^2} \| D^s J_{\varepsilon} u \|_{L^2} \\
\lesssim \| (\partial_x u)^2 \|_{H^{s-1}} \| u \|_{H^s} \\
\lesssim \| u \|^3_{H^s}.
\]
(3.68)

**Estimating the nonlocal** $D^{s-2} \partial_x J_{\varepsilon} (\rho^2)$. To estimate the fourth integral of (3.59), we apply the Cauchy-Schwarz inequality to obtain
\[
\left| \int_T D^{s-2} \partial_x J_{\varepsilon} (\rho^2) D^s J_{\varepsilon} u \, dx \right| \leq \| D^{s-2} \partial_x J_{\varepsilon} (\rho^2) \|_{L^2} \| D^s J_{\varepsilon} u \|_{L^2} \\
\lesssim \| \rho^2 \|_{H^{s-1}} \| u \|_{H^s} \\
\lesssim \| u \|_{H^s} \| \rho \|^2_{H^{s-1}}.
\]
(3.69)

Combining (3.66)-(3.69) we obtain the differential inequality
\[
\frac{1}{2} \frac{d}{dt} \| J_{\varepsilon} u(t) \|^2_{H^s} \lesssim \| u \|^3_{H^s} + \| u \|_{H^s} \| \rho \|^2_{H^{s-1}}.
\]
(3.70)

Now, applying the operator $D^{s-1}$ to both sides of (3.58), multiplying the resulting equation by $D^{s-1} J_{\varepsilon} \rho$, and integrating over the torus gives
\[
\frac{1}{2} \frac{d}{dt} \| J_{\varepsilon} \rho \|^2_{H^{s-1}} = - \int_T D^{s-1} J_{\varepsilon} \partial_x (u \rho) D^{s-1} J_{\varepsilon} \rho \, dx.
\]
(3.71)

To bound (3.71) we first commute the $J_{\varepsilon}$ over to the $J_{\varepsilon} \rho$. Then we commute $D^{s-1} \partial_x$
and \( u \) to obtain
\[
\int_{\mathcal{T}} D^{s-1} J_\varepsilon \partial_x (u \rho) D^{s-1} J_\varepsilon \rho \, dx = \int_{\mathcal{T}} [D^{s-1} \partial_x, u] \rho D^{s-1} J_\varepsilon J_\varepsilon \rho \, dx \tag{3.72}
\]
\[
+ \int_{\mathcal{T}} u D^{s-1} \partial_x \rho D^{s-1} J_\varepsilon J_\varepsilon \rho \, dx. \tag{3.73}
\]

For (3.72) we utilize the Cauchy-Schwarz inequality and Lemma 6 to achieve
\[
\left| \int_{\mathcal{T}} [D^{s-1} \partial_x, u] \rho D^{s-1} J_\varepsilon J_\varepsilon \rho \, dx \right| \leq \| [D^{s-1} \partial_x, u] \rho \|_{L^2} \| D^{s-1} J_\varepsilon J_\varepsilon \rho \|_{L^2} \nonumber
\]
\[
\lesssim \| u \|_{H^s} \| \rho \|_{H^{s-1}}^2. \nonumber
\]

For (3.73) we use integration by parts and the Cauchy-Schwarz inequality to obtain
\[
\left| \int_{\mathcal{T}} u D^{s-1} \partial_x \rho D^{s-1} J_\varepsilon J_\varepsilon \rho \, dx \right| \approx \left| \int_{\mathcal{T}} \partial_x u D^{s-1} \rho D^{s-1} J_\varepsilon J_\varepsilon \rho \, dx \right| \nonumber
\]
\[
\lesssim \| \partial_x u \|_{L^\infty} \| \rho \|_{H^{s-1}}^2. \nonumber
\]

Combining estimates for (3.72) and (3.73) we obtain the following differential inequality
\[
\frac{1}{2} \frac{d}{dt} \| J_\varepsilon \rho(t) \|_{H^{s-1}}^2 \lesssim \| u \|_{H^s} \| \rho \|_{H^{s-1}}^2. \nonumber
\]

Assuming that \( \| J_\varepsilon u(t) \|_{H^s} \) and \( \| J_\varepsilon \rho(t) \|_{H^{s-1}} \) are not zero, then we have
\[
\frac{d}{dt} \| J_\varepsilon u(t) \|_{H^s} \lesssim \| u \|_{H^s}^2 + \| \rho \|_{H^{s-1}}^2 \tag{3.74}
\]
\[
\frac{d}{dt} \| J_\varepsilon \rho(t) \|_{H^{s-1}} \lesssim \| u \|_{H^s} \| \rho \|_{H^{s-1}} \tag{3.75}
\]
Adding (3.74) and (3.75) we obtain

\[ \frac{d}{dt} \|(J_{\varepsilon} u(t), J_{\varepsilon} \rho(t))\|_s \lesssim \|(u, \rho)\|_s^2. \]

Finally, using our lifespan estimate (2.3) gives us

\[ |F'(t)| = \left| \frac{d}{dt} \|(J_{\varepsilon} u(t), J_{\varepsilon} \rho(t))\|_s \right| \leq c_s \|(u_0, \rho_0)\|_s^2. \]

Therefore, we have shown $F(t)$ is Lipschitz. This concludes our proof of Lemma 14 which implies the completion of Proposition 2. \qed

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3.3 UNIQUENESS

**Proposition 3.** (Uniqueness) For initial data \((u_0, \rho_0) \in H^s \times H^{s-1}, \ s > 5/2\), The Cauchy problem \((2.1)\) has a unique solution in the space \(C([0, T]; H^s \times H^{s-1})\).

**Proof.** Let \((u_0, \rho_0) \in H^s \times H^{s-1}\) and let \((u, \rho)\) and \((w, \phi)\) be two solutions to the Cauchy problem for the 2-component CH system with \(u(x,0) = u_0 = w(x,0)\) and \(\rho(x,0) = \rho_0 = \phi(x,0)\). That is to say

\[
\begin{align*}
\begin{cases}
\partial_t u + u \partial_x u + F(u, \rho) = 0, \\
\partial_t \rho + \partial_x (u \rho) = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
\partial_t w + w \partial_x w + F(w, \phi) = 0, \\
\partial_t \phi + \partial_x (w \phi) = 0
\end{cases}
\end{align*}
\tag{3.76}
\]

**Lemma 16.** We have that \(v(t)\) and \(\theta(t)\) satisfy the following differential inequalities

\[
\quad \frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\gamma}^2 \lesssim (\|u\|_{H^s} + \|w\|_{H^s}) \|v\|_{H^\gamma}^2 \\
\quad + (\|\rho\|_{H^{s-1}} + \|\phi\|_{H^{s-1}}) \|\theta\|_{H^{\gamma-1}} \|v\|_{H^\gamma}
\tag{3.77}
\]

\[
\quad \frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}}^2 \lesssim \|f\|_{H^\gamma} \|\theta\|_{H^{\gamma-1}}^2 + \|h\|_{H^{s-1}} \|v\|_{H^\gamma} \|\theta\|_{H^{\gamma-1}}
\tag{3.78}
\]

where \(f = u + w\) and \(h = \rho + \phi\).

**Proof.** Subtracting the second set of equations of \((3.76)\) from the first set and setting the differences \(v = u - w\) and \(\theta = \rho - \phi\) and the quantities \(f = u + w\) and
\[ h = \rho + \phi \] the Cauchy problem for \((v, \theta)\) is given by

\[
\begin{align*}
\partial_t v &= -\frac{1}{2} \partial_x (fv) - [F(u, \rho) - F(w, \phi)], \\
\partial_t \theta &= -\frac{1}{2} \partial_x (f\theta + vh) = -\frac{1}{2} [\partial_x (f\theta) + \partial_x (vh)] \\
v(0) &= 0 \text{ and } \theta(0) = 0
\end{align*}
\]

Fix \(\gamma \in (1/2, s - 2)\) with \(\gamma < 1\). Calculating the \(H^\gamma\) energy of \(v\) gives us the equation

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\gamma}^2 = -\frac{1}{2} \int_T D^\gamma \partial_x (fv) D^\gamma v dx - \int_T D^\gamma [F(u, \rho) - F(w, \theta)] D^\gamma v dx \quad (3.79)
\]

To bound the first term on the right-hand side of (3.79) we commute \(D^\gamma \partial_x\) with \(f\) via Lemma 7.

Thus, we have

\[
\left| \int_T D^\gamma \partial_x (fv) D^\gamma v dx \right| \leq \left| \int_T [D^\gamma \partial_x, f] v D^\gamma v dx \right| + \left| \int_T f D^\gamma \partial_x v D^\gamma v dx \right| \quad (3.80)
\]

For the first term on the right hand side of (3.80) we apply Lemma 7 to obtain

\[
\left| \int_T [D^\gamma \partial_x, f] v D^\gamma v dx \right| \lesssim \|f\|_{H^s} \|v\|_{H^\gamma}^2 \quad (3.81)
\]

For the second term of (3.80) we use integration by parts to obtain

\[
\left| \int_T f D^\gamma \partial_x v D^\gamma v dx \right| \approx \left| \int_T f \partial_x (D^\gamma v)^2 dx \right| \leq \|\partial_x f\|_{L^\infty} \left| \int_T (D^\gamma v)^2 dx \right| \lesssim \|f\|_{H^s} \|v\|_{H^\gamma}^2.
\]

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For the non-local term of (3.79), applying the Cauchy-Schwarz inequality we get

\[ \left| \int_T D^\gamma [F(u, \rho) - F(w, \phi)] D^\gamma v dx \right| \lesssim \| F(u, \rho) - F(w, \theta) \|_{H^\gamma} \| v \|_{H^\gamma}. \]

Using Lemma 4 we have

\[ \| F(u, \rho) - F(w, \phi) \|_{H^\gamma} \lesssim (\| u \|_{H^{\gamma+1}} + \| w \|_{H^{\gamma+1}}) \| v \|_{H^\gamma} + (\| \rho \|_{H^\gamma} + \| \phi \|_{H^\gamma}) \| \theta \|_{H^{\gamma-1}} \]

Combining the above terms, we have our \( H^{\gamma} \) energy estimate for \( v(t) \) as shown in (3.77) of Lemma 16.

For the \( H^{\gamma-1} \) energy for \( \theta \) we have

\[
\frac{1}{2} \frac{d}{dt} \| \theta(t) \|_{H^{\gamma-1}}^2 = -\frac{1}{2} \int_T D^{\gamma-1} \partial_x (f \theta) D^{\gamma-1} \theta dx \tag{3.82}
- \frac{1}{2} \int_T D^{\gamma-1} \partial_x (vh) D^{\gamma-1} \theta dx \tag{3.83}
\]

To bound (3.82) and (3.83) we will make use of Lemma 7 and the Cauchy-Schwarz inequality.

**Estimation of (3.82):**

\[
-\frac{1}{2} \int_T D^{\gamma-1} \partial_x (f \theta) D^{\gamma-1} \theta dx = -\frac{1}{2} \int_T [D^{\gamma-1} \partial_x, f] \theta D^{\gamma-1} \theta dx
- \frac{1}{2} \int_T f D^{\gamma-1} \partial_x \theta D^{\gamma-1} \theta dx \lesssim \| f \|_{H^{\gamma}} \| \theta \|_{H^{\gamma-1}}^2.
\]
Estimation of (3.83):

$$-\frac{1}{2} \int_T D^{\gamma-1} \partial_x (v h) D^{\gamma-1} \partial_x \theta \, dx \lesssim \|v h\|_{H^\gamma} \|\theta\|_{H^{\gamma-1}} \lesssim \|h\|_{H^{s-1}} \|v\|_{H^\gamma} \|\theta\|_{H^{\gamma-1}}. \quad (3.84)$$

Combining terms (3.82) to (3.83) we have our $H^\gamma$ energy estimate for $\theta(t)$ as stated in (3.78). This concludes our proof of Lemma 16. □

From our lifespan estimate (2.3) we have $\|u\|_{H^s} + \|w\|_{H^s}, \|\rho\|_{H^{s-1}} + \|\phi\|_{H^{s-1}} \lesssim 1$, $\|f\|_{H^s} \lesssim 1$, and $\|h\|_{H^{s-1}} \lesssim 1$. Thus our system from Lemma 16 becomes

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^\gamma}^2 \lesssim \|v\|_{H^\gamma}^2 + \|\theta\|_{H^{\gamma-1}} \|v\|_{H^\gamma} \quad (3.85)$$

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}}^2 \lesssim \|\theta\|_{H^{\gamma-1}}^2 + \|v\|_{H^\gamma} \|\theta\|_{H^{\gamma-1}}. \quad (3.86)$$

Carrying out the differentiation on the left hand sides of (3.85) and (3.86), simplifying and setting $x = \|v(t)\|_{H^\gamma}$ and $y = \|\theta(t)\|_{H^{\gamma-1}}$, we obtain the following system

$$\frac{dx}{dt} \lesssim x + y \quad (3.87)$$

$$\frac{dy}{dt} \lesssim x + y. \quad (3.88)$$

Adding (3.87) and (3.88) and setting $z = x + y$ we have

$$\frac{dz}{dt} \leq c_s z. \quad (3.89)$$

Solving (3.89) gives us

$$z(t) \leq e^{c_s t} z(0) = 0.$$
This implies that
\[ \|(u - w, \rho - \phi)\|_s \leq 0. \]

So we have that \( \|u - w\|_{H^\gamma} = 0 \) and \( \|\rho - \phi\|_{H^{\gamma-1}} = 0 \). Thus, \( u = w \) and \( \rho = \phi \).
3.4 CONTINUOUS DEPENDENCE ON INITIAL DATA

Proposition 4. The solution map for the 2-component Camassa-Holm system i.v.p. (2.1) from $H^s \times H^{s-1}$ → $C(I; H^s \times H^{s-1})$ given by $(u_0, \rho_0) \rightarrow (u, \rho)$ is continuous.

Proof. Fix $(u_0, \rho_0) \in H^s \times H^{s-1}$ and let $\{(u_{0,n}, \rho_{0,n})\} \subset H^s \times H^{s-1}$ be a sequence such that $\lim_{n \rightarrow \infty} (u_{0,n}, \rho_{0,n}) = (u_0, \rho_0)$. Then if $(u, \rho)$ is the solution to the 2-component CH-system i.v.p. with initial data $(u_0, \rho_0)$ and if $(u_n, \rho_n)$ is the solution to the 2-component CH-system i.v.p. with initial data $(u_{0,n}, \rho_{0,n})$, we will demonstrate that

$$\lim_{n \rightarrow \infty} (u_n, \rho_n) = (u, \rho), \text{ in } C(I; H^s \times H^{s-1}).$$

Let $\eta > 0$. We need to show that there exists $N > 0$ such that

$$\forall n > N \implies \|u - u_n\|_{C(I; H^s)} + \|\rho - \rho_n\|_{C(I; H^{s-1})} < \eta.$$ 

We will be using energy estimates in the $H^s$ and $H^{s-1}$ norms to get around the difficulty of estimating the Burgers terms. Thus, we will use the $J_\varepsilon$ convolution operator to smooth out the initial data. Let $\varepsilon \in (0, 1]$. We take $(u^\varepsilon, \rho^\varepsilon)$ to be the solution to the 2-component CH-system i.v.p. with initial data $(J_\varepsilon u_0, J_\varepsilon \rho_0)$ and $(u^\varepsilon_n, \rho^\varepsilon_n)$ be the solution with initial data $(J_\varepsilon u_{0,n}, J_\varepsilon \rho_{0,n})$. Applying the triangle inequality, we arrive
at
\[ \|u - u_n\|_{C(I;H^s)} + \|\rho - \rho_n\|_{C(I;H^{s-1})} \leq \|u - u^\varepsilon\|_{C(I;H^s)} + \|u^\varepsilon - u_n^\varepsilon\|_{C(I;H^s)} + \|u_n^\varepsilon - u_n\|_{C(I;H^s)} + \|\rho - \rho^\varepsilon\|_{C(I;H^{s-1})} + \||u^\varepsilon - u_n^\varepsilon|_{C(I;H^{s-1})| + \||\rho^\varepsilon - \rho_n^\varepsilon|_{C(I;H^{s-1})|.} \]
\[ (3.90) \]

We will prove that each of these terms can be bounded by $\eta/6$ for suitable choices of $\varepsilon$ and $N$. We note that this $\varepsilon$ parameter we have introduced will be independent of $N$ and will only depend on $\eta$ whereas the choice of $N$ will depend on both $\eta$ and $\varepsilon$.

**Estimation of** $\|u^\varepsilon - u_n^\varepsilon\|_{C(I;H^s)} + \|\rho^\varepsilon - \rho_n^\varepsilon\|_{C(I;H^{s-1})}$. We begin with these terms of (3.90) because all terms under the norms have been smoothed by $J_\varepsilon$. This fact allows us to directly bound this term via $H^s$ and $H^{s-1}$ norms.

**Lemma 17.** For a constant $c_s$ only depending on $s$, we have the following bound for
\[ \|u^\varepsilon - u_n^\varepsilon\|_{C(I;H^s)} + \|\rho^\varepsilon - \rho_n^\varepsilon\|_{C(I;H^{s-1})} \]
\[ \|(u^\varepsilon(t) - u_n^\varepsilon(t), \rho^\varepsilon(t) - \rho_n^\varepsilon(t))\|_s \leq e^{\frac{c_s t}{\varepsilon}} \|((u^\varepsilon(0) - u_n^\varepsilon(0), \rho^\varepsilon(0) - \rho_n^\varepsilon(0))\|_s. \]

**Proof.** We begin by setting the differences $v$ and $\theta$ to be
\[ v = u^\varepsilon - u_n^\varepsilon \text{ and } \theta = \rho^\varepsilon - \rho_n^\varepsilon. \]
A direct calculation verifies that \((v, \theta)\) solves the Cauchy problem

\[
\begin{align*}
\partial_t v &= -u^\varepsilon (\partial_x v) - (\partial_x u^\varepsilon_n) v - D^{-2} \partial_x \left[ f v + \frac{\sigma}{2} h \theta \right] - \frac{1}{2} D^{-2} \partial_x \left[ \partial_x f \partial_x v \right] \\
\partial_t \theta &= -\frac{1}{2} \left[ \partial_x(f \theta) + \partial_x(h \theta) \right] \\
v(0) &= u^\varepsilon(0) - u^\varepsilon_n(0) = J_\varepsilon u_0 - J_\varepsilon u_{0,n} \\
\theta(0) &= \rho^\varepsilon(0) - \rho^\varepsilon_n(0) = J_\varepsilon \rho_0 - J_\varepsilon \rho_{0,n}.
\end{align*}
\] (3.91)

where \(f = u^\varepsilon + u^\varepsilon_n\) and \(h = \rho^\varepsilon + \rho^\varepsilon_n\).

Applying the operator \(D^s\) to both sides of the first line in (3.91), multiplying by \(D^sv\) and finally integrating over the torus yeilds the \(H^s\) energy estimate

\[
\frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 = - \int_T D^s [u^\varepsilon(\partial_x v)] D^s v dx - \int_T D^s [(\partial_x u^\varepsilon_n) v] D^s v dx
\]

\[
- \int_T D^{s-2} \partial_x [(u^\varepsilon + u^\varepsilon_n) v] D^s v dx
\]

\[
- \frac{1}{2} \int_T D^{s-2} \partial_x [(\partial_x u^\varepsilon + \partial_x u^\varepsilon_n) \partial_x v] D^s v dx
\]

\[
- \frac{\sigma}{2} \int_T D^{s-2} \partial_x [(\rho^\varepsilon + \rho^\varepsilon_n) \theta] D^s v dx.
\] (3.92 - 3.95)

Estimating (3.92). For the first term of (3.92) we have

\[
\left| \int_T D^s [u^\varepsilon(\partial_x v)] D^s v dx \right| \leq \left| \int_T [D^s, u^\varepsilon] \partial_x v D^s v dx \right| + \left| \int_T u^\varepsilon D^s \partial_x v D^s v dx \right|.
\] (3.96)

For the first term of (3.96) we use the Cauchy-Schwarz inequality and the Kato-Ponce...
lemma to obtain
\[
\left| \int_T [D^s, u^\varepsilon] \partial_x v D^s v dx \right| \leq \| [D^s, u^\varepsilon] \partial_x v \|_{L^2} \| D^s v \|_{L^2} \\
\lesssim \left( \| D^s u^\varepsilon \|_{L^2} \| \partial_x v \|_{L^\infty} + \| \partial_x u^\varepsilon \|_{L^\infty} \| D^{s-1} \partial_x v \|_{L^2} \right) \| v \|_{H^s} \\
\lesssim \| u^\varepsilon \|_{H^s} \| v \|_{H^s}^2.
\]

For the second term in (3.96), we integrate by parts and use the Cauchy-Schwarz inequality to yield
\[
\left| \int_T u^\varepsilon D^s \partial_x v D^s v dx \right| \approx \left| \int_T \partial_x u^\varepsilon (D^s v)^2 dx \right| \\
\leq \| \partial_x u^\varepsilon \|_{L^\infty} \int_T (D^s v)^2 dx \\
\lesssim \| u^\varepsilon \|_{H^s} \| v \|_{H^s}^2.
\]

Now, for the second term of (3.92) we have
\[
\left| \int_T D^s [u_n^\varepsilon v] D^s v dx \right| \leq \left| \int_T D^s, v \, \partial_x u_n^\varepsilon D^s v dx \right| + \left| \int_T D^s \partial_x u_n^\varepsilon D^s v dx \right|. \tag{3.97}
\]

For the first term of (3.97), we use the Cauchy-Schwarz inequality and the Kato-Ponce lemma to obtain
\[
\left| \int_T [D^s, v] \partial_x u_n^\varepsilon D^s v dx \right| \leq \| [D^s, v] \partial_x u_n^\varepsilon \|_{L^2} \| D^s v \|_{L^2} \\
\lesssim \left( \| D^s v \|_{L^2} \| \partial_x u_n^\varepsilon \|_{L^\infty} + \| \partial_x v \|_{L^\infty} \| D^{s-1} \partial_x u_n^\varepsilon \|_{L^2} \right) \| v \|_{H^s}^2 \\
\lesssim \| u_n^\varepsilon \|_{H^s} \| v \|_{H^s}^2.
\]

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For the second term of (3.97) we integrate by parts and use the Cauchy-Schwarz inequality to yield

\[
\left| \int_T v D^s \partial_x u_n^\epsilon D^s v dx \right| \leq \|v\|_{L^\infty} \int_T |D^s \partial_x u_n^\epsilon D^s v| dx \\
\lesssim \|u_n^\epsilon\|_{H^{s+1}} \|v\|^2_{H^s} \\
\lesssim \frac{c_s}{\epsilon} \|v\|^2_{H^s},
\]

where the last line follows from Lemma 48. Therefore, we find that

(3.92) \lesssim \left( \|u^\epsilon\|_{H^s} + \frac{c_s}{\epsilon} \right) \|v\|^2_{H^s}.

**Estimating (3.93).** We use the Cauchy-Schwarz inequality and the algebra property to obtain

\[
\left| \int_T D^{s-2} \partial_x [(u^\epsilon + u_n^\epsilon) v] D^s v dx \right| \lesssim (\|u^\epsilon\|_{H^s} + \|u_n^\epsilon\|_{H^s}) \|v\|^2_{H^s}.
\]

**Estimating (3.94).** We use the Cauchy-Schwarz inequality and the algebra property to yield

\[
\left| \int_T D^{s-2} \partial_x [(\partial_x u^\epsilon + \partial_x u_n^\epsilon) \partial_x v] D^s v dx \right| \lesssim (\|u^\epsilon\|_{H^s} + \|u_n^\epsilon\|_{H^s}) \|v\|^2_{H^s}.
\]

**Estimating (3.95).** Again, we use the Cauchy-Schwarz inequality and the algebra
Combining estimates (3.92)-(3.95) together and we yield the following $H^s$ energy estimate for $v$

$$\frac{1}{2} \frac{d}{dt} \|v\|_{H^s}^2 \lesssim \left( \|u^\epsilon\|_{H^s} + \|u_n^\epsilon\|_{H^s} + \frac{c_s}{\epsilon} \right) \|v\|_{H^s}^2$$

$$+ (\|\rho^\epsilon\|_{H^{s-1}} + \|\rho_n^\epsilon\|_{H^{s-1}}) \|\theta\|_{H^{s-1}} \|v\|_{H^s}.$$

Now, by applying the $D^{s-1}$ operator to both sides of the second line in (3.91), multiplying by $D^{s-1} \theta$ and integrating over the torus yields the following $H^{s-1}$ energy estimate for $\theta$

$$\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^{s-1}}^2 = -\frac{1}{2} \int_T D^{s-1} \partial_x (f \theta) D^{s-1} \theta dx$$

$$- \frac{1}{2} \int_T D^{s-1} \partial_x (v \theta) D^{s-1} \theta dx$$

(3.98)

(3.99)

**Estimating (3.98).** Here we utilize Lemma 7 and the Cauchy-Schwarz inequality to obtain

$$-\frac{1}{2} \int_T D^{s-1} \partial_x (f \theta) D^{s-1} \theta dx = -\frac{1}{2} \int_T [D^{s-1} \partial_x, f] \theta D^{s-1} \theta dx$$

$$- \frac{1}{2} \int_T f D^{s-1} \partial_x \theta D^{s-1} \theta dx \lesssim \|f\|_{H^s} \|\theta\|_{H^{s-1}}^2.$$

**Estimating (3.99).** Again we utilize Lemma 7 and the Cauchy-Schwarz inequality
to achieve

\[-\frac{1}{2} \int T D^{s-1} \partial_x (vh) D^{s-1} \theta dx = -\frac{1}{2} \int T [D^{s-1} \partial_x, v] h D^{s-1} \theta dx \]

\[-\frac{1}{2} \int T v D^{s-1} \partial_x h D^{s-1} \theta dx \]

\[
\lesssim ||h||_{H^s} ||v||_{H^s} ||\theta||_{H^{s-1}}^2
\]

\[
\lesssim \frac{c_s}{\varepsilon} ||v||_{H^s} ||\theta||_{H^{s-1}},
\]

where the last line follows from Lemma 48. Therefore, by combining our estimates for (3.98) and (3.99) we have the following $H^{s-1}$ energy estimate for $\theta$

\[
\frac{1}{2} \frac{d}{dt} ||\theta||_{H^{s-1}}^2 \lesssim ||f||_{H^s} ||\theta||_{H^{s-1}}^2 + \frac{c_s}{\varepsilon} ||v||_{H^s} ||\theta||_{H^{s-1}}.
\]

Thus, our system of nonlinear ODEs is

\[
\frac{1}{2} \frac{d}{dt} ||v(t)||_{H^s}^2 \lesssim \left( ||u^\varepsilon||_{H^s} + ||u_n^\varepsilon||_{H^s} + \frac{c_s}{\varepsilon} \right) ||v||_{H^s}^2
\]

\[
+ (||\rho^\varepsilon||_{H^{s-1}} + ||\rho_n^\varepsilon||_{H^{s-1}}) ||\theta||_{H^{s-1}} ||v||_{H^s}
\]

\[
\frac{1}{2} \frac{d}{dt} ||\theta(t)||_{H^{s-1}}^2 \lesssim ||f||_{H^s} ||\theta||_{H^{s-1}}^2 + \frac{c_s}{\varepsilon} ||v||_{H^s} ||\theta||_{H^{s-1}}.
\]

Now, we use our lifespan estimate from (2.3) to deduce that $||u^\varepsilon||_{H^s} + ||u_n^\varepsilon||_{H^s} \lesssim 1,$
\[ \|\rho^\varepsilon\|_{H^{s-1}} + \|\rho_n^\varepsilon\|_{H^{s-1}} \lesssim 1, \text{ and } \|f\|_{H^{s}} \lesssim 1. \]

So our system becomes

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{s}}^2 \leq \frac{c_s}{\varepsilon}(\|v\|_{H^{s}}^2 + \|\theta\|_{H^{s-1}} \|v\|_{H^{s}}) \tag{3.100}
\]

\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{s-1}}^2 \leq \frac{c_s}{\varepsilon}(\|\theta\|_{H^{s-1}}^2 + \|v\|_{H^{s}} \|\theta\|_{H^{s-1}}). \tag{3.101}
\]

Carrying out the differentiation on the left hand sides of both (3.100) and (3.101), simplifying and letting \(x = \|v\|_{H^{s}}\) and \(y = \|\theta\|_{H^{s-1}}\) gives us the system

\[
\frac{dx}{dt} \leq \frac{c_s}{\varepsilon}(x + y) \tag{3.102}
\]

\[
\frac{dy}{dt} \leq \frac{c_s}{\varepsilon}(x + y). \tag{3.103}
\]

Adding (3.102) and (3.103) together and setting \(z = x + y\) gives us

\[
\frac{dz}{dt} \leq \frac{c_s}{\varepsilon}z. \tag{3.104}
\]

Solving (3.104) we obtain

\[ z(t) \leq e^{\frac{c_s}{\varepsilon}t}z(0). \]

This implies that

\[ \|(u^\varepsilon(t) - u_n^\varepsilon(t), \rho^\varepsilon(t) - \rho_n^\varepsilon(t))\|_{s} \leq e^{\frac{c_s}{\varepsilon}t}(\|(u^\varepsilon(0) - u_n^\varepsilon(0), \rho^\varepsilon(0) - \rho_n^\varepsilon(0))\|_{s}). \]

which completes the proof of Lemma 17. \(\square\)
Now, we can take $N$ sufficiently large such that

$$\|(u^\varepsilon(0) - u_n^\varepsilon(0), \rho^\varepsilon(0) - \rho_n^\varepsilon(0))\|_s \leq e^{-\frac{\varepsilon^2 t}{3} \eta^3}.$$ 

Therefore, we have that

$$\|(u^\varepsilon(t) - u_n^\varepsilon(t), \rho^\varepsilon(t) - \rho_n^\varepsilon(t))\|_s \leq \frac{\eta^3}{3}.$$  

**Estimation of** $\|u^\varepsilon - u\|_{C(I; H^s)} + \|\rho^\varepsilon - \rho\|_{C(I; H^{s-1})}$ **and** $\|u_n^\varepsilon - u_n\|_{C(I; H^s)} + \|\rho_n^\varepsilon - \rho_n\|_{C(I; H^{s-1})}$.

We begin by setting the differences $v$ and $v_n$ to be

$$v = u^\varepsilon - u \text{ and } v_n = u_n^\varepsilon - u_n.$$  

Furthermore, we set the differences $\theta$ and $\theta_n$ to be

$$\theta = \rho^\varepsilon - \rho \text{ and } \theta_n = \rho_n^\varepsilon - \rho_n.$$  

In fact, $(v, \theta)$ and $(v_n, \theta_n)$ will satisfy the same energy estimates so we will adopt the notation of a subscript in parentheses, i.e. $v_{(n)}$ and $\theta_{(n)}$, to mean that an equation holds both with and without the subscript. We observe that $(v_{(n)}, \theta_{(n)})$ solves the
Cauchy problem

\[
\partial_t v_{(n)} = v_{(n)} \partial_x v_{(n)} - u_{(n)}^\varepsilon \partial_x v_{(n)} - v_{(n)} \partial_x u_{(n)}^\varepsilon \\
- D^{-2} \partial_x \left[ 2u_{(n)}^\varepsilon v_{(n)} - v_{(n)}^2 \right] \\
- D^{-2} \partial_x \left[ \partial_x u_{(n)}^\varepsilon \partial_x v_{(n)} - \frac{1}{2}(\partial_x v_{(n)})^2 \right] \\
- \frac{\sigma}{2} D^{-2} \partial_x \left[ 2\rho_{(n)}^\varepsilon \theta_{(n)} - \theta_{(n)}^2 \right] 
\]

(3.105)

\[
\partial_t \theta_{(n)} = -\frac{1}{2} \left[ \partial_x (f \theta_{(n)}) + \partial_x (hv_{(n)}) \right] 
\]

(3.106)

\[
v(0) = u_{(n)}^\varepsilon(0) - u_{(n)}(0) = J_\varepsilon u_0_{(n)} - u_0_{(n)} \\
\theta(0) = \rho_{(n)}^\varepsilon(0) - \rho_0(0) = J_\varepsilon \rho_0_{(n)} - \rho_0_{(n)}.
\]

where \( f = u_{(n)}^\varepsilon + u_{(n)} \) and \( h = \rho_{(n)}^\varepsilon + \rho_{(n)} \).

We will now examine the \( H^s \) energy of \( v_{(n)} \).

**Lemma 18.** \( v_{(n)} \) satisfies the following differential inequality

\[
\frac{1}{2} \frac{d}{dt} \| v_{(n)}(t) \|^2_{H^s} \lesssim \| v_{(n)} \|^3_{H^s} + \| u_{(n)}^\varepsilon \|_{H^s} \| v_{(n)} \|_{H^s}^2 + \| v_{(n)} \|_{H^{s-1}} \| u_{(n)}^\varepsilon \|_{H^{s+1}} \| v_{(n)} \|_{H^s} \\
+ \| \theta_{(n)} \|^2_{H^{s-1}} \| v_{(n)} \|_{H^s} + \| \rho_{(n)}^\varepsilon \|_{H^{s-1}} \| \theta_{(n)} \|_{H^{s+1}} \| v_{(n)} \|_{H^s}. 
\]

(3.107)

**Proof.** We will apply the operator \( D^s \) to both sides of (3.105), multiply by \( D^s v_{(n)} \)
and integrate over the torus to obtain

\[
\frac{1}{2} \frac{d}{dt} \| v(n) \|^2_{H^s} = \int_T D^s(v(n) \partial_x v(n))D^s v(n) dx \\
- \int_T D^s(u^\varepsilon_n \partial_x v(n))D^s v(n) dx \\
- \int_T D^s(v(n) \partial_x u^\varepsilon_n)D^s v(n) dx \\
- 2 \int_T D^{s-2} \partial_x (u^\varepsilon_n v(n))D^s v(n) dx \\
+ \int_T D^{s-2} \partial_x (v^2_n)D^s v(n) dx \\
- \int_T D^{s-2} \partial_x [\partial_x u^\varepsilon_n \partial_x v(n)] D^s v(n) dx \\
+ \frac{1}{2} \int_T D^{s-2} \partial_x [(\partial_x v(n))^2] D^s v(n) dx \\
- \sigma \int_T D^{s-2} \partial_x (\rho^\varepsilon_n \theta(n)) D^s v(n) dx \\
+ \frac{\sigma}{2} \int_T D^{s-2} \partial_x (\theta^2_n) D^s v(n) dx \\
\approx E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 + E_9.
\]

To bound each of these terms we make repeated use of the Kato-Ponce lemma, the Cauchy-Schwarz inequality, the Sobolev Embedding Theorem, and integration by parts.

**Estimate for** \(E_1\):

\[
E_1 = \int_T D^s(v(n) \partial_x v(n))D^s v(n) dx \\
= \int_T [D^s, v(n)] \partial_x v(n) D^s v(n) dx + \int_T v(n) D^s \partial_x v(n) D^s v(n) dx \lesssim \| v(n) \|^3_{H^s}.
\]
Estimate for $E_2$: 

$$E_2 = - \int_T D^s (u_n^\varepsilon \partial_x v_n) D^s v_n \, dx$$

$$= - \int_T \left[ D^s, u_n^\varepsilon \right] \partial_x v_n D^s v_n \, dx - \int_T u_n^\varepsilon D^s \partial_x v_n D^s v_n \, dx \lesssim \|u_n^\varepsilon\|_{H^s} \|v_n\|_{H^s}^2.$$

Estimate for $E_3$: 

$$E_3 = - \int_T D^s (v_n \partial_x u_n^\varepsilon) D^s v_n \, dx$$

$$= - \int_T \left[ D^s, v_n \right] \partial_x u_n^\varepsilon D^s v_n \, dx - \int_T v_n D^s \partial_x u_n^\varepsilon D^s v_n \, dx$$

$$\lesssim \|u_n^\varepsilon\|_{H^s} \|v_n\|_{H^s}^2 + \|v_n\|_{H^{s-1}} \|u_n^\varepsilon\|_{H^{s+1}} \|v_n\|_{H^s}.$$

Estimate for $E_4$: 

$$E_4 = -2 \int_T D^{s-2} \partial_x (u_n^\varepsilon v_n) D^s v_n \, dx \lesssim \|u_n^\varepsilon\|_{H^s} \|v_n\|_{H^s}^2.$$

Estimate for $E_5$: 

$$E_5 = \int_T D^{s-2} \partial_x (v_n^2) D^s v_n \, dx \lesssim \|v_n\|_{H^s}^3.$$

Estimate for $E_6$: 

$$E_6 = - \int_T D^{s-2} \partial_x \left[ \partial_x u_n^\varepsilon \partial_x v_n \right] D^s v_n \, dx \lesssim \|u_n^\varepsilon\|_{H^s} \|v_n\|_{H^s}^2.$$
Estimate for $E_7$:

$$E_7 = \frac{1}{2} \int_T D^{s-2} \partial_x \left[ (\partial_x v_n)^2 \right] D^s v_n \, dx \lesssim \|v_n\|_{H^s}^3.$$ 

Estimate for $E_8$:

$$E_8 = -\sigma \int_T D^{s-2} \partial_x (\rho_n^\xi \theta_n) D^s v_n \, dx \lesssim \|\rho_n^\xi\|_{H^{s-1}} \|\theta_n\|_{H^{s-1}} \|v_n\|_{H^s}.$$ 

Estimate for $E_9$:

$$E_9 = \sigma \int_T D^{s-2} \partial_x (\theta_n^2 \theta_n) D^s v_n \, dx \lesssim \|\theta_n\|_{H^{s-1}} \|v_n\|_{H^s}.$$ 

Combining estimates $E_1$ to $E_9$, we have the differential inequality (3.107). This concludes our proof of Lemma 18. 

**Lemma 19.** $\theta_n$ satisfies the differential inequality

$$\frac{1}{2} \frac{d}{dt} \|\theta_n(t)\|_{H^{s-1}}^2 \lesssim \|f\|_{H^s} \|\theta_n\|_{H^{s-1}}^2 \|h\|_{H^{s-1}} \|v_n\|_{H^s} \|\theta_n\|_{H^{s-1}}$$

$$+ \|v_n\|_{H^s} \|\theta_n\|_{H^{s-1}}^2 + \|v_n\|_{H^{s-1}} \|\rho_n^\xi\|_{H^s} \|\theta_n\|_{H^{s-1}}$$

(3.108)

**Proof.** Applying the operator $D^{s-1}$ to (3.106), multiplying the aforementioned
by $D^{s-1}\theta(n)$ and integrating over the torus we obtain

$$
\frac{1}{2} \frac{d}{dt} \|\theta(n)\|_{H^{s-1}}^2 = -\frac{1}{2} \int_T D^{s-1} \partial_x (f\theta(n)) D^{s-1} \theta(n) dx
\quad - \frac{1}{2} \int_T D^{s-1} \partial_x (v(n)h) D^{s-1} \theta(n) dx
\quad \geq M_1 + M_2.
$$

To bound each of these terms we make use of Lemma 7, the Cauchy-Schwarz inequality, the Sobolev Embedding Theorem, and integration by parts.

**Estimate for $M_1$:**

$$
M_1 = -\frac{1}{2} \int_T D^{s-1} \partial_x (f\theta(n)) D^{s-1} \theta(n) dx
\quad = -\frac{1}{2} \int_T \left[ D^{s-1} \partial_x, f \right] \theta(n) D^{s-1} \theta(n) dx - \frac{1}{2} \int_T f D^{s-1} \partial_x \theta(n) D^{s-1} \theta(n) dx
\quad \lesssim \|f\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2.
$$

**Estimate for $M_2$:**

$$
M_2 = -\frac{1}{2} \int_T D^{s-1} \partial_x (v(n)h) D^{s-1} \theta(n) dx
\quad = -\frac{1}{2} \int_T \left[ D^{s-1} \partial_x, v(n) \right] h D^{s-1} \theta(n) dx - \frac{1}{2} \int_T v(n) D^{s-1} \partial_x h D^{s-1} \theta(n) dx
\quad \lesssim \|h\|_{H^{s-1}} \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}} + \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2 + \|v(n)\|_{H^{s-1}} \|\rho\|_{H^s} \|\theta(n)\|_{H^{s-1}},
$$

which is obtained by seeing that $h = -\theta(n) + 2\rho\phi(n)$. Combining estimates $M_1$ and $M_2$ we acquire the differential inequality in Lemma
which completes our proof. □

Now, we have that (3.107) and (3.108) give us the following system of ODEs

\[
\frac{1}{2} \frac{d}{dt} \|v(n)(t)\|_{H^s}^2 \lesssim \|v(n)\|_{H^s}^3 + \|u(\epsilon)\|_{H^s} \|v(n)\|_{H^s}^2 + \|v(n)\|_{H^{s-1}} \|u(\epsilon)\|_{H^{s+1}} \|v(n)\|_{H^s} \\
+ \|\theta(n)\|_{H^{s-1}}^2 \|v(n)\|_{H^s} + \|\rho(\xi(\epsilon))\|_{H^{s-1}} \|\theta(n)\|_{H^{s-1}} \|v(n)\|_{H^s}. \quad (3.109)
\]

\[
\frac{1}{2} \frac{d}{dt} \|\theta(n)(t)\|_{H^{s-1}}^2 \lesssim \|f\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2 + \|h\|_{H^{s-1}} \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}} \\
+ \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2 + \|v(n)\|_{H^{s-1}} \|\rho(\xi(\epsilon))\|_{H^s} \|\theta(n)\|_{H^{s-1}}. \quad (3.110)
\]

From our solution size estimate we have \(\|u(\epsilon)\|_{H^s}, \|\rho(\xi)\|_{H^{s-1}}, \|f\|_{H^s}, \text{ and } \|h\|_{H^{s-1}}\) are bounded by a constant. Furthermore, by using our well-posedness estimate and Lemma 48 we can take \(\|v(n)\|_{H^s}, \|\theta(n)\|_{H^{s-1}} \lesssim 1\) and \(\|u(\epsilon)\|_{H^{s+1}}, \|\rho(\xi)\|_{H^s} \leq c_s/\varepsilon\). This implies that (3.109) and (3.110) give us the following simpler system of ODEs

\[
\frac{1}{2} \frac{d}{dt} \|v(n)(t)\|_{H^s}^2 \lesssim \|v(n)\|_{H^s}^3 + \|v(n)\|_{H^s}^2 + \frac{c_s}{\varepsilon} \|v(n)\|_{H^{s-1}} \|v(n)\|_{H^s} \\
+ \|\theta(n)\|_{H^{s-1}}^2 \|v(n)\|_{H^s} + \|\theta(n)\|_{H^{s-1}} \|v(n)\|_{H^s}, \quad (3.111)
\]

\[
\frac{1}{2} \frac{d}{dt} \|\theta(n)(t)\|_{H^{s-1}}^2 \lesssim \|\theta(n)\|_{H^{s-1}}^2 + \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}} \\
+ \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2 + \frac{c_s}{\varepsilon} \|v(n)\|_{H^{s-1}} \|\theta(n)\|_{H^{s-1}}. \quad (3.112)
\]

We now make several observations that will simplify (3.111) and (3.112). First, we shall use the Interpolation Lemma 23 with \(s_1 = \gamma\) and \(s_2 = s\) to obtain an \(H^{s-1}\)
estimate on \( v(n) \). Here we assume \( \gamma \in (1/2, 1) \) and find that

\[
\|v(n)\|_{H^{s-1}} \lesssim \|v(n)\|_{H^s}^{\frac{1}{s-\gamma}} \|v(n)\|_{H^s}^{\frac{1}{s-\gamma}} \lesssim \|v(n)\|_{H^s} \|v(n)(0)\|_{H^s}^{1-\frac{1}{s-\gamma}} \lesssim \|v(n)\|_{H^s}^{\frac{1}{s-\gamma}}.
\]

(3.113)

Next, we will verify that

\[
\|v(n)\|_{H^s} = o(\varepsilon^{s-\gamma}).
\]

(3.114)

Noting observations (3.113) and (3.114), we may reduce our system (3.111) and (3.112) to the following

\[
\frac{1}{2} \frac{d}{dt} \|v(n)(t)\|_{H^s}^2 \lesssim \|v(n)\|_{H^s}^2 + \|\theta(n)\|_{H^{s-1}}^2 \|v(n)\|_{H^s}^2 + \|\theta(n)\|_{H^{s-1}} \|v(n)\|_{H^s}.
\]

(3.115)

\[
\frac{1}{2} \frac{d}{dt} \|\theta(n)(t)\|_{H^{s-1}}^2 \lesssim \|\theta(n)\|_{H^{s-1}}^2 + \|v(n)\|_{H^s} \|\theta(n)\|_{H^{s-1}}^2 + \|v(n)\|_{H^s}^2 \|\theta(n)\|_{H^{s-1}}^2 + \|\theta(n)\|_{H^{s-1}}^2.
\]

(3.116)

where \( \delta = \delta(\varepsilon) \) and \( \delta \to 0 \) as \( \varepsilon \to 0 \). We continue with a proof of the claim made in (3.114).
Lemma 20. For \( t \in I \), the \( H^\gamma \) norm of \( v(n) \) satisfies

\[
\| v(n) \|_{H^\gamma} = o(\varepsilon^{s-\gamma}).
\] (3.117)

Proof: Calculating the \( H^\gamma \) energy of \( v(n) \), we have

\[
\frac{1}{2} \frac{d}{dt} \| v(n) \|^2_{H^\gamma} = \int_\mathbb{T} D^\gamma(v(n) \partial_x v(n)) D^\gamma v(n) \, dx \\
- \int_\mathbb{T} D^\gamma(u^\varepsilon(n) \partial_x v(n)) D^\gamma v(n) \, dx \\
- \int_\mathbb{T} D^\gamma(v(n) \partial_x u^\varepsilon(n)) D^\gamma v(n) \, dx \\
- 2 \int_\mathbb{T} D^{\gamma-2} \partial_x(u^\varepsilon(n) v(n)) D^\gamma v(n) \, dx \\
+ \int_\mathbb{T} D^{\gamma-2} \partial_x(v^2(n)) D^\gamma v(n) \, dx \\
- \int_\mathbb{T} D^{\gamma-2} \partial_x [\partial_x u^\varepsilon(n) \partial_x v(n)] \, D^\gamma v(n) \, dx \\
+ \frac{1}{2} \int_\mathbb{T} D^{\gamma-2} \partial_x [(\partial_x v(n))^2] \, D^\gamma v(n) \, dx \\
- \sigma \int_\mathbb{T} D^{\gamma-2} \partial_x (\rho^\varepsilon(n) \theta(n)) \, D^\gamma v(n) \, dx \\
+ \frac{\sigma}{2} \int_\mathbb{T} D^{\gamma-2} \partial_x (\theta_0^2(n)) \, D^\gamma v(n) \, dx
\]

\[= F_1 + F_2 + F_3 + F_4 + F_5 + F_6 + F_7 + F_8 + F_9. \]

To bound each of these terms we make repeated use of the Kato-Ponce lemma, the Cauchy-Schwarz inequality, the Sobolev Embedding Theorem, Negative Multiplier Lemma \( \S \) and integration by parts.
Estimate for $F_1$:

$$F_1 = \int_T D^\gamma (v_n \partial_x v_n) D^\gamma v_n dx$$

$$= \int_T [D^\gamma, v_n] \partial_x v_n D^\gamma v_n dx + \int_T v_n D^\gamma \partial_x v_n D^\gamma v_n dx \lesssim \|v_n\|_{H^\gamma}^2.$$  

Estimate for $F_2$:

$$F_2 = -\int_T D^\gamma (u^\epsilon_n \partial_x v_n) D^\gamma v_n dx$$

$$= -\int_T [D^\gamma, u^\epsilon_n] \partial_x v_n D^\gamma v_n dx - \int_T u^\epsilon_n D^\gamma \partial_x v_n D^\gamma v_n dx \lesssim \|v_n\|_{H^\gamma}^2.$$  

Estimate for $F_3$:

$$F_3 = -\int_T D^\gamma (v_n \partial_x u^\epsilon_n) D^\gamma v_n dx$$

$$= -\int_T [D^\gamma, v_n] \partial_x u^\epsilon_n D^\gamma v_n dx - \int_T v_n D^\gamma \partial_x u^\epsilon_n D^\gamma v_n dx$$

$$\lesssim \|v_n\|_{H^\gamma}^2.$$  

Estimate for $F_4$:

$$F_4 = -2 \int_T D^{\gamma-2} \partial_x (u^\epsilon_n v_n) D^\gamma v_n dx \lesssim \|v_n\|_{H^\gamma}^2.$$  

Estimate for $F_5$:

$$F_5 = \int_T D^{\gamma-2} \partial_x (v^2_n) D^\gamma v_n dx \lesssim \|v_n\|_{H^\gamma}^2.$$
Estimate for $F_6$:

$$F_6 = - \int T D^{\gamma-2} \partial_x \left[ \partial_x u^\varepsilon(n) \partial_x v(n) \right] D^\gamma v(n) dx \lesssim \|v(n)\|_{H^\gamma}^2.$$ 

Estimate for $F_7$:

$$F_7 = \frac{1}{2} \int T D^{\gamma-2} \partial_x (\partial_x v(n))^2 D^\gamma v(n) dx \lesssim \|v(n)\|_{H^\gamma}^2.$$ 

Estimate for $F_8$:

$$F_8 = -\sigma \int T D^{\gamma-2} \partial_x (\rho^\varepsilon(n) \theta(n)) D^\gamma v(n) dx \lesssim \|\theta(n)\|_{H^{\gamma-1}} \|v(n)\|_{H^\gamma}.$$ 

Estimate for $F_9$:

$$F_9 = \frac{\sigma}{2} \int T D^{\gamma-2} \partial_x (\theta^2(n)) D^\gamma v(n) dx \lesssim \|\theta(n)\|_{H^{\gamma-1}} \|v(n)\|_{H^\gamma}.$$ 

Calculating an $H^{\gamma-1}$ norm for $\theta(n)$ we have

$$\frac{1}{2} \frac{d}{dt} \|\theta(n)\|_{H^{\gamma-1}}^2 = -\frac{1}{2} \int T D^{\gamma-1} \partial_x (f \theta(n)) D^{\gamma-1} \theta(n) dx$$

$$-\frac{1}{2} \int T D^{\gamma-1} \partial_x (v(n) h) D^{\gamma-1} \theta(n) dx$$

$$\lesssim G_1 + G_2.$$ 

To bound each of these terms we make use of Lemma 7 the Cauchy-Schwarz
inequality, the Sobolev Embedding Theorem, and integration by parts.

**Estimate for \(G_1\):

\[
G_1 = -\frac{1}{2} \int_T D^{\gamma-1} \partial_x (f \theta_n) D^{\gamma-1} \theta_n \, dx = -\frac{1}{2} \int_T \left[ D^{\gamma-1} \partial_x, f \right] \theta_n D^{\gamma-1} \theta_n \, dx - \frac{1}{2} \int_T f D^{\gamma-1} \partial_x \theta_n D^{\gamma-1} \theta_n \, dx \lesssim \| \theta_n \|^2_{H^{\gamma-1}}.
\]

**Estimate for \(G_2\):

\[
G_2 = -\frac{1}{2} \int_T D^{\gamma-1} \partial_x (v_n h) D^{\gamma-1} \theta_n \, dx = -\frac{1}{2} \int_T \left[ D^{\gamma-1} \partial_x, h \right] v_n D^{\gamma-1} \theta_n \, dx - \frac{1}{2} \int_T h D^{\gamma-1} \partial_x v_n D^{\gamma-1} \theta_n \, dx \lesssim \| v_n \|_{H^{\gamma}} \| \theta_n \|_{H^{\gamma-1}}.
\]

Combining estimates \(F_1\) to \(F_9\) and \(G_1\) to \(G_2\), we obtain the following system

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \| v_n \|^2_{H^{\gamma}} & \lesssim \| v_n \|^2_{H^{\gamma}} + \| v_n \|_{H^{\gamma}} \| \theta_n \|_{H^{\gamma-1}} \\
\frac{1}{2} \frac{d}{dt} \| \theta_n \|_{H^{\gamma-1}} & \lesssim \| \theta_n \|^2_{H^{\gamma-1}} + \| v_n \|_{H^{\gamma}} \| \theta_n \|_{H^{\gamma-1}}
\end{align*}
\tag{3.118}
\]

Carrying out differentiation on the left hand sides of (3.118), simplifying and
setting \( x = \| v(n)(t) \|_{H^\gamma} \) and \( y = \| \theta(n)(t) \|_{H^{\gamma-1}} \) gives us the following system

\[
\begin{align*}
\frac{dx}{dt} & \lesssim x + y \quad (3.119) \\
\frac{dy}{dt} & \lesssim x + y. \quad (3.120)
\end{align*}
\]

Adding (3.119) and (3.120) together and setting \( z = x + y \) delivers us the ODE

\[
\frac{d}{dt}(x + y) \lesssim x + y \implies \frac{dz}{dt} \leq c_s z \quad (3.121)
\]

Solving the ODE (3.121) we obtain

\[
z(t) \leq e^{c_s t} z(0) \implies \|(v(n), \theta(n))\|_{\gamma} \leq e^{c_s t} \|(v(n)(0), \theta(n)(0))\|_{\gamma}. \quad (3.122)
\]

By Lemma 11, this implies that

\[
\|(v(n), \theta(n))\|_{\gamma} = o(\varepsilon^{\delta-\gamma}),
\]

which concludes our proof. \( \Box \)

Returning to the issue of our system of inequalities (3.115) and (3.116), by letting \( z = \|(v(n), \theta(n))\|_s \) we obtain

\[
\frac{dz}{dt} \lesssim z^2 + z + \delta. \quad (3.123)
\]

The quadratic expression \( z^2 + z + \delta \) has roots

\[
r_{-1} = \frac{-1 - \sqrt{1 - 4\delta}}{2} \quad \text{and} \quad r_0 = \frac{-1 + \sqrt{1 - 4\delta}}{2}.
\]
We now put our first restriction on $\varepsilon$ so that this expression has real roots. We observe that both $r_0$ and $r_{-1}$ are negative and as $\delta \to 0$ we have $r_{-1} \to -1$ and $r_0 \to 0$. Setting $R = \sqrt{1 - 4\delta}$, we proceed to solve (3.123) as follows

$$\left( \frac{1}{z - r_0} - \frac{1}{z - r_{-1}} \right) dz \lesssim R dt.$$}

Integrating the above inequality gives

$$\ln|z - r_0|_0^t - \ln|z - r_{-1}|_0^t \lesssim Rt.$$

Reinserting the missing constant $c_s$, this simplifies to

$$\frac{z(t) - r_0}{zt - r_{-1}} \leq \beta, \quad \text{where} \quad \beta \doteq e^{c_s RT} \left( \frac{z(0) - r_0}{z(0) - r_{-1}} \right). \quad (3.124)$$

We observe that the right hand side of $\beta$ is a function of $\varepsilon$ as

$$z(0) = \|j_\varepsilon * u_{0,(n)} - u_{0,(n)}\|_{H^s} + \|j_\varepsilon * \rho_{0,(n)} - \rho_{0,(n)}\|_{H^{s-1}}.$$

Now we must observe two cases.

Case of $z = \|v\|_{H^s} + \|\theta\|_{H^{s-1}}$: Using Lemma 11, we have that

$$z(0) = \|(j_\varepsilon * u_0 - u_0, j_\varepsilon * \rho_0 - \rho_0)\|_s \to 0 \text{ as } \varepsilon \to 0.$$
From (3.124) we obtain

\[ z(t) \leq z(t) - r_0 \leq \beta[z(t) - r_{-1}] . \]

Solving for \( z(t) \) gives us

\[ z(t) \leq \frac{-r_{-1}}{1 - \beta} \beta \rightarrow 0 , \quad \text{as} \quad \beta \rightarrow 0 . \] (3.125)

Therefore, for sufficiently small \( \varepsilon \) we can bound \( \| (u^\varepsilon - u, \rho^\varepsilon - \rho) \|_s \) by \( \eta/3 \).

**Case of** \( z = \| v_n \|_{H^s} + \| \theta_n \|_{H^{s-1}} \): We begin by bounding \( z(0) \) by noticing

\[
\| j_\varepsilon * u_{0,n} - u_{0,n} \|_{H^s} \leq \| j_\varepsilon * u_{0,n} - j_\varepsilon * u_0 \|_{H^s} \\
+ \| j_\varepsilon * u_0 - u_0 \|_{H^s} \\
+ \| u_0 - u_{0,n} \|_{H^s} \\
\leq 2 \| u_{0,n} - u_0 \|_{H^s} + \| j_\varepsilon * u_0 - u_0 \|_{H^s}
\]

and

\[
\| j_\varepsilon * \rho_{0,n} - \rho_{0,n} \|_{H^{s-1}} \leq \| j_\varepsilon * \rho_{0,n} - j_\varepsilon * \rho_0 \|_{H^{s-1}} \\
+ \| j_\varepsilon * \rho_0 - \rho_0 \|_{H^{s-1}} \\
+ \| \rho_0 - \rho_{0,n} \|_{H^{s-1}} \\
\leq 2 \| \rho_{0,n} - \rho_0 \|_{H^{s-1}} + \| j_\varepsilon * \rho_0 - \rho_0 \|_{H^{s-1}} .
\]
Then we can bound $\beta$ as

$$
\beta = e^{c_s RT} \left( \frac{z(0) - r_0}{z(0) - r_{-1}} \right) \leq e^{c_s RT} \left( z(0) - r_0 \right)
\leq \frac{e^{c_s RT}}{r_{-1}} \left( 2\|u_{0,n} - u_0\|_{H^s} + \|j_{\varepsilon} * u_0 - u_0\|_{H^s} + 2\|\rho_{0,n} - \rho_0\|_{H^{s-1}} + \|j_{\varepsilon} * \rho_0 - \rho_0\|_{H^{s-1}} \right)
+ \frac{r_0 e^{c_s RT}}{r_{-1}}.
$$

(3.126)

Using (3.126) we may choose $\varepsilon$ sufficiently small and $N$ sufficiently large such that $\beta < 1/2$. Then proceeding along the same lines as (3.125) we can obtain

$$
\frac{-r_{-1}}{1 - \beta} \beta \leq 2\beta.
$$

(3.127)

Using (3.126) and (3.127) we may further refine the choice of $\varepsilon$ and $N$ so that $z(t) \leq \eta/3$, which completes this case.

Finally, we observe that these two cases along with Lemma 17 allows us to bound (3.90) by $\eta$, thereby completing our proof of continuity of the flow map.
CHAPTER 4
NON-UNIFORM DEPENDENCE ON THE TORUS

In this section, we will prove Theorem 2 which reads

**Theorem 7.** If $s > 5/2$ then the data-to-solution map for the 2-component Camassa-Holm system defined by the Cauchy problem (2.1) is not uniformly continuous from any bounded subset of $H^s \times H^{s-1}$ into $C([0, T]; H^s \times H^{s-1})$.

It suffices to show that there exists two sequences of solutions $(u_n, \rho_n)$ and $(w_n, \phi_n)$ in $C([0, T]; H^s \times H^{s-1})$ such that

$$\| (u_n, \rho_n) \|_s + \| (w_n, \phi_n) \|_s \lesssim 1$$

$$\lim_{n \to \infty} \| (u_n(0) - w_n(0), \rho_n(0) - \phi_n(0)) \|_s = 0,$$

and

$$\liminf_{n \to \infty} \| (u_n(t) - w_n(t), \rho_n(t) - \phi_n(t)) \|_s \gtrsim |\sin t|, \quad 0 \leq t \leq T < 1.$$

**Approximate 2-Component CH solutions.** We consider the approximate solutions of the form

$$u^{\omega,n}(x, t) = \rho^{\omega,n}(x, t) = \omega n^{-1} + n^{-s} \cos(n x - \omega t). \quad (4.1)$$
where \( n \in \mathbb{Z}^+ \) and \( \omega = \pm 1 \). Substituting (4.1) into the 2-component CH system generates the following expressions for the errors \( E \) and \( F \):

\[
E \doteq \partial_t u^{\omega,n} + u^{\omega,n} \partial_x u^{\omega,n} + (1 - \partial_x^2)^{-1} \partial_x \left[ (u^{\omega,n})^2 + \frac{1}{2} (\partial_x u^{\omega,n})^2 + \frac{\sigma}{2} (\rho^{\omega,n})^2 \right]
\]

\[
F \doteq \partial_t \rho^{\omega,n} + \partial_x (u^{\omega,n} \rho^{\omega,n}).
\]

To estimate the \( H^\gamma \) norm of these errors, we use the following fact. For \( \gamma \in \mathbb{R}, n \in \mathbb{Z}^+ \) and \( n \gg 1 \), we have

\[
\| \cos(nx - \alpha) \|_{H^{\gamma}} \approx n^\gamma, \quad \alpha \in \mathbb{R}.
\] (4.2)

The relation also holds if cosine is replaced by sine. Hence, for \( s \geq 0 \) and \( n \gg 1 \) we have

\[
\| u^{\omega,n}(t) \|_{H^{\gamma}} = \| \rho^{\omega,n}(t) \|_{H^{\gamma}} = \| \omega n^{-1} + n^{-s} \cos(nx - \omega t) \|_{H^{\gamma}} \lesssim n^{-1} + n^{-s+\gamma}.
\]

**Lemma 21.** For \( n \gg 1 \) and \( s > (1 + \gamma)/2 \),

\[
\| E(t) \|_{H^{\gamma}} \lesssim n^{-r_s} \quad (4.3)
\]

\[
\| F(t) \|_{H^{\gamma-1}} \lesssim n^{-s-1+\gamma} \quad (4.4)
\]
where \( r_s > 0 \) and

\[
r_s = \begin{cases} 
2s - 1 - \gamma, & \text{if } (1 + \gamma)/2 < s \leq 3 \\
s + 2 - \gamma, & \text{if } s \geq 3
\end{cases}
\]

**Proof.** First we note the following derivatives of our approximate solutions

\[
\partial_t u_{\omega,n}(x,t) = \omega n^{-s} \sin(nx - \omega t)
\]

\[
\partial_x u_{\omega,n}(x,t) = -n^{-s+1} \sin(nx - \omega t)
\]

\[
\partial^2_x u_{\omega,n}(x,t) = -n^{-s+2} \cos(nx - \omega t).
\]

Let \( E_B \) denote the Burgers term portion of the error \( E \) and \( E_{nl} \) denote the nonlocal portion. Then we have

\[
E_B = \omega n^{-s} \sin(nx - \omega t) - \omega n^{-s} \sin(nx - \omega t) - \frac{1}{2} n^{-2s+1} \sin 2(nx - \omega t)
\]

\[
= -\frac{1}{2} n^{-2s+1} \sin 2(nx - \omega t)
\]

and

\[
E_{nl} = D^{-2}[-2\omega n^{-s} \sin(nx - \omega t) - n^{-2s+1} \sin 2(nx - \omega t)] + \frac{1}{2} D^{-2}[n^{-2s+3} \sin 2(nx - \omega t)]
\]

\[
- \sigma D^{-2} \left[ \omega n^{-s} \sin(nx - \omega t) + \frac{1}{2} n^{-2s+1} \sin 2(nx - \omega t) \right].
\]
By taking $E = E_B + E_{nl}$ we have that

$$E = -\frac{1}{2}n^{-2s+1} \sin 2(nx - \omega t) + D^{-2}[-2\omega n^{-s} \sin(n x - \omega t) - n^{-2s+1} \sin 2(nx - \omega t)]$$
$$+ \frac{1}{2} D^{-2}[n^{-2s+3} \sin 2(nx - \omega t)] - \sigma D^{-2} \left[\omega n^{-s} \sin(n x - \omega t) + \frac{1}{2}n^{-2s+1} \sin 2(nx - \omega t)\right]$$

and

$$F = -\omega n^{-s} \sin(n x - \omega t) - n^{-2s+1} \sin 2(nx - \omega t).$$

Taking the $H^\gamma$ norm of $E$ and the $H^{\gamma-1}$ norm of $F$ and applying the triangle inequality yields

$$\|E(t)\|_{H^\gamma} \lesssim n^{-2s+1+\gamma} + n^{-s-2+\gamma} + n^{-2s-1+\gamma} \lesssim n^{-2s+1+\gamma} + n^{-s-2+\gamma}$$
$$\|F(t)\|_{H^{\gamma-1}} \lesssim n^{-s-1+\gamma} + n^{-2s+\gamma} \lesssim n^{-s-1+\gamma}.$$

Thus completing our proof. □

**Actual 2-Component CH solutions.** Let $(u_{\omega,n}(x,t), \rho_{\omega,n}(x,t))$ be the solution to the 2-Component CH system i.v.p. (2.1) with the initial data given by the approximate solution $(u^{\omega,n}(x,0), \rho^{\omega,n}(x,0))$; i.e., $(u_{\omega,n}(x,t), \rho_{\omega,n}(x,t))$ solves the Cauchy problem

$$\partial_t u_{\omega,n} + u_{\omega,n} \partial_x u_{\omega,n} + (1 - \partial_x^2)^{-1} \partial_x \left[\left(u_{\omega,n}\right)^2 + \frac{1}{2}(\partial_x u_{\omega,n})^2 + \frac{\sigma}{2}(\rho_{\omega,n})^2\right] = 0 \quad (4.5)$$
$$\partial_t \rho_{\omega,n} + \partial_x (u_{\omega,n} \rho_{\omega,n}) = 0 \quad (4.6)$$
$$u_{\omega,n}(x,0) = \rho_{\omega,n}(x,0) = \omega n^{-1} + n^{-s} \cos(nx). \quad (4.7)$$
Note that (4.2) implies the initial data \((u^{\omega,n}(x,0), \rho^{\omega,n}(x,0)) \in H^s \times H^{s-1}\) for all \(s \geq 0\), since
\[
\|u_{\omega,n}(x,0)\|_{H^s} = \|\omega n^{-1} + n^{-s} \cos(nx)\|_{H^s} \lesssim n^{-1} + 1
\]
\[
\|\rho_{\omega,n}(x,0)\|_{H^{s-1}} = \|\omega n^{-1} + n^{-s} \cos(nx)\|_{H^{s-1}} \lesssim n^{-1}.
\]
Hence, by Theorem 1, there exists a \(T > 0\) such that, for \(n >> 1\), the Cauchy problem (2.1) has a unique solution in \(C([0,T]; H^s \times H^{s-1}) \cap C^1([0,T]; H^s \times H^{s-1})\) with lifespan \(T > 0\) such that \((u^{\omega,n}, \rho^{\omega,n})\) satisfies (2.3) for \(t \in [0,T]\).

To estimate the difference between the approximate and actual solutions, we note that \(v = u^{\omega,n} - u_{\omega,n}\) and \(\theta = \rho^{\omega,n} - \rho_{\omega,n}\) satisfies the following initial value problem
\[
\begin{align*}
\partial_t v &= E - \frac{1}{2} \partial_x (fv) - D^{-2} \partial_x \left[ (fv) + \frac{1}{2} \partial_x f \partial_x v + \frac{\sigma}{2} h \theta \right] \tag{4.8} \\
\partial_t \theta &= F - \frac{1}{2} \partial_x (vh + f\theta). \tag{4.9}
\end{align*}
\]
where \(v(x,0) = \theta(x,0) = 0\), \(x \in \mathbb{T}, t \in \mathbb{R}, f = u^{\omega,n} + u_{\omega,n},\) and \(h = \rho^{\omega,n} + \rho_{\omega,n}\). Furthermore, \(E\) and \(F\) satisfy the estimates (4.3) and (4.4), respectively.

**Lemma 22.** Let \(s > 5/2\). The differences \(v\) and \(\theta\) satisfy
\[
\|(v(t), \theta(t))\|_\gamma \lesssim n^{-rs} + n^{-s-1+\gamma} \lesssim n^{-\varepsilon_s}, \quad 0 \leq t \leq T, \quad n >> 1,
\]
where
\[
\varepsilon_s = s + 1 - \gamma,
\]
$r_s > 0$ is given by (4.3), and $\gamma$ is chosen so that $3/2 < \gamma + 1 < s$ and $\gamma < 1$.

**Proof.** We will prove Lemma 22 by finding energy estimates for the differences $v$ and $\theta$. Applying the operator $D^\gamma$ to both sides of (4.8), multiplying by $D^\gamma v$ and integrating over the torus gives us

\[
\frac{1}{2} \frac{d}{dt} \| v \|_{H^\gamma}^2 = \int_T D^\gamma E D^\gamma v dx \tag{4.10}
\]

\[- \frac{1}{2} \int_T D^\gamma \partial_x (fv) D^\gamma v dx \tag{4.11}
\]

\[- \int_T D^{\gamma-2} \partial_x (fv) D^\gamma v dx \tag{4.12}
\]

\[- \frac{1}{2} \int_T D^{\gamma-2} \partial_x [\partial_x f \partial_x v] D^\gamma v dx \tag{4.13}
\]

\[- \frac{1}{2} \int_T D^{\gamma-2} \partial_x (h\theta) D^\gamma v dx. \tag{4.14}
\]

**Estimation of (4.10):** Using the Cauchy-Schwarz inequality we obtain

\[
\int_T D^\gamma E D^\gamma v dx \leq \| E \|_{H^\gamma} \| v \|_{H^\gamma} \lesssim n^{-r_s} \| v \|_{H^\gamma}.
\]

**Estimation of (4.11):** Here we will commute $D^\gamma \partial_x$ and $f$ and apply the triangle inequality to obtain

\[
\left| - \frac{1}{2} \int_T D^\gamma \partial_x (fv) D^\gamma v dx \right| \leq \frac{1}{2} \int_T [D^\gamma \partial_x, f] v D^\gamma v dx + \frac{1}{2} \int_T f D^\gamma \partial_x v D^\gamma v dx \tag{4.15}
\]

To estimate the first integral in (4.15) we will use the Cauchy-Schwarz inequality and
Lemma 7 to achieve

\[ \frac{1}{2} \int_T [D^\gamma \partial_x, f] v D^\gamma v dx \lesssim \|[D^\gamma \partial_x, f] v\|_{L^2} \|D^\gamma v\|_{L^2} \lesssim \|f\|_{H^{s}} \|v\|_{H^{\gamma}}. \]

To bound the second integral in (4.15) we integrate by parts and use the Cauchy-Schwarz inequality to obtain

\[ \frac{1}{2} \int_T f D^\gamma \partial_x v D^\gamma v dx \lesssim \|\partial_x f\|_{L^\infty} \|v\|_{H^{\gamma}} \lesssim \|f\|_{H^{s}} \|v\|_{H^{\gamma}}. \]

**Estimation of (4.12):** Given that \((D^s f, g)_{L^2} = (f, D^s g)_{L^2}\), we use the Cauchy-Schwarz inequality to get

\[ -\frac{1}{2} \int_T \partial_x (fv) D^\gamma v dx \lesssim \|f\|_{L^\infty} \|v\|_{H^{\gamma}} \lesssim \|f\|_{H^{s}} \|v\|_{H^{\gamma}}. \]

**Estimation of (4.13):** Using Lemma 2 with \(\gamma \in (\frac{1}{2}, 1)\) and the Cauchy-Schwarz inequality we have

\[ -\frac{1}{2} \int_T \partial_x [D^\gamma [\partial_x f \partial_x v]] D^\gamma v dx \lesssim \|[D^\gamma [\partial_x f \partial_x v]]\|_{L^2} \|D^\gamma v\|_{L^2} \lesssim \|\partial_x f \partial_x v\|_{H^{\gamma-1}} \|v\|_{H^{\gamma}} \lesssim \|f\|_{H^{s+1}} \|v\|_{H^{\gamma}}. \]
**Estimation of (4.14):** Here we use the Cauchy-Schwarz inequality to get

\[
\left| -\frac{\sigma}{2} \int_{T} D^{\gamma-2} \partial_x(h\theta) D^{\gamma} v dx \right| \lesssim \|h\|_{H^{\gamma}} \|\theta\|_{H^{\gamma-1}} \|v\|_{H^{\gamma}}.
\]

Combining estimates (4.10) to (4.14) we arrive at the following differential inequality for \(v(t)\)

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{\gamma}}^2 \lesssim \eta^{-s} \|v\|_{H^{\gamma}} + \|f\|_{H^{\gamma}} \|v\|_{H^{\gamma}}^2 + \|f\|_{H^{\gamma+1}} \|v\|_{H^{\gamma}} + \|h\|_{H^{\gamma}} \|\theta\|_{H^{\gamma-1}} \|v\|_{H^{\gamma}}. \tag{4.16}
\]

Using the Sobolev Embedding Theorem in conjunction with our solution size estimate (2.3), we have that \(\|f\|_{H^{\gamma}}, \|f\|_{H^{\gamma+1}}, \text{ and } \|h\|_{H^{\gamma}} \lesssim 1\), which reduces (4.16) to

\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{\gamma}}^2 \lesssim \eta^{-s} \|v\|_{H^{\gamma}} + \|v\|_{H^{\gamma}}^2 + \|\theta\|_{H^{\gamma-1}} \|v\|_{H^{\gamma}}. \tag{4.17}
\]

Now applying the operator \(D^{\gamma-1}\) to both sides of (4.9), multiplying by \(D^{\gamma-1}\theta\) and integrating over the torus gives us the following energy for \(\theta\)

\[
\frac{1}{2} \frac{d}{dt} \|\theta\|_{H^{\gamma-1}}^2 = \int_{\mathbb{T}} D^{\gamma-1} FD^{\gamma-1} \theta dx \tag{4.18}
\]

\[
- \frac{1}{2} \int_{\mathbb{T}} D^{\gamma-1} \partial_x(hv) D^{\gamma-1} \theta dx \tag{4.19}
\]

\[
- \frac{1}{2} \int_{\mathbb{T}} D^{\gamma-1} \partial_x(f\theta) D^{\gamma-1} \theta dx. \tag{4.20}
\]
**Estimation of (4.18):** Using the Cauchy-Schwarz inequality we obtain

\[
\left| \int_T D^{\gamma-1} F D^{\gamma-1} \theta dx \right| \lesssim \| F \|_{H^{\gamma-1}} \| \theta \|_{H^{\gamma-1}} \lesssim n^{-s-1+\gamma} \| \theta \|_{H^{\gamma-1}}.
\]

**Estimation of (4.19):** We use the Cauchy-Schwarz inequality and the algebra property to yield

\[
\left| -\frac{1}{2} \int_T D^{\gamma-1} \partial_x (hv) D^{\gamma-1} \theta dx \right| \lesssim \| h \|_{H^{\gamma}} \| v \|_{H^{\gamma}} \| \theta \|_{H^{\gamma-1}}.
\]

**Estimation of (4.20):** We will commute \( D^{\gamma-1} \partial_x \) and \( f \) and apply the triangle inequality to obtain

\[
\left| -\frac{1}{2} \int_T D^{\gamma-1} \partial_x (f \theta) D^{\gamma-1} \theta dx \right| \leq \left| \frac{1}{2} \int_T [D^{\gamma-1} \partial_x, f] \theta D^{\gamma-1} \theta dx \right| + \left| \frac{1}{2} \int_T f D^{\gamma-1} \partial_x \theta D^{\gamma-1} \theta dx \right|.
\]

(4.21)

To estimate the first integral in (4.21) we use the Cauchy-Schwarz inequality and Lemma 7 to get

\[
\left| \frac{1}{2} \int_T [D^{\gamma-1} \partial_x, f] \theta D^{\gamma-1} \theta dx \right| \lesssim \|[D^{\gamma-1} \partial_x, f] \theta \|_{L^2} \| D^{\gamma-1} \theta \|_{L^2}
\]

\[
\lesssim \| f \|_{H^s} \| \theta \|_{H^{\gamma-1}}^2.
\]

To estimate the second integral in (4.21) we integrate by parts and use the Cauchy-Schwarz inequality to obtain

\[
\left| \frac{1}{2} \int_T f D^{\gamma-1} \partial_x \theta D^{\gamma-1} \theta dx \right| \lesssim \| \partial_x f \|_{L^\infty} \| \theta \|_{H^{\gamma-1}}^2.
\]
Combining estimates (4.18) to (4.20) we obtain the following differential inequality for $\theta$

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}}^2 \lesssim n^{-s-1+\gamma} \|\theta\|_{H^{\gamma-1}}^2 + \|h\|_{H^{\gamma}} \|v\|_{H^{\gamma}} \|\theta\|_{H^{\gamma-1}} + \|f\|_{H^{s}}^2 + \|\partial_x f\|_{L^\infty} \|\theta\|_{H^{\gamma-1}}^2,$$

(4.22)

Using the Sobolev Embedding Theorem and our solution size estimate (2.3) we can bound $\|h\|_{H^{\gamma}}, \|f\|_{H^{s}},$ and $\|\partial_x f\|_{L^\infty}$ by a constant, which reduces the above differential inequality to

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma}}^2 \lesssim n^{-s-1+\gamma} \|\theta\|_{H^{\gamma-1}} + \|v\|_{H^{\gamma}} \|\theta\|_{H^{\gamma-1}} + \|\theta\|_{H^{\gamma-1}}^2.$$

The differential inequalities for $v$ and $\theta$ now give us the following system of ODEs

$$\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{\gamma}}^2 \lesssim n^{-r_s} \|v\|_{H^{\gamma}} + \|v\|_{H^{\gamma}}^2 + \|\theta\|_{H^{\gamma-1}} \|v\|_{H^{\gamma}}$$

(4.23)

$$\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}}^2 \lesssim n^{-s-1+\gamma} \|\theta\|_{H^{\gamma-1}} + \|v\|_{H^{\gamma}} \|\theta\|_{H^{\gamma-1}} + \|\theta\|_{H^{\gamma-1}}^2.$$

(4.24)

Carrying out differentiation on the right hand side of both (4.23) and (4.24) and simplifying gives us

$$\frac{d}{dt} \|v(t)\|_{H^{\gamma}} \lesssim n^{-r_s} + \|v\|_{H^{\gamma}} + \|\theta\|_{H^{\gamma-1}}$$

(4.25)

$$\frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}} \lesssim n^{-s-1+\gamma} + \|v\|_{H^{\gamma}} + \|\theta\|_{H^{\gamma-1}}.$$

(4.26)

Letting $x = \|v\|_{H^{\gamma}}$ and $y = \|\theta\|_{H^{\gamma-1}}$ and adding (4.25) and (4.26) delivers us the
following ODE
\[ \frac{d}{dt}(x + y) \lesssim n^{-r_s} + n^{-s-1+\gamma} + x + y. \]

Now let \( z = x + y \) and we have
\[ \frac{dz}{dt} \lesssim n^{-r_s} + n^{-s-1+\gamma} + z. \]

Solving the above differential inequality we have
\[ \frac{d}{dt}[e^{-t}z] \lesssim (n^{-r_s} + n^{-s-1+\gamma})e^{-t} \implies z(t) \lesssim -(n^{-r_s} + n^{-s-1+\gamma}) + z(0)e^t. \]

Since \( z(0) = 0 \) we have that
\[ z(t) \lesssim n^{-r_s} + n^{-s-1+\gamma} \implies \| (v, \theta) \|_\gamma \lesssim n^{-r_s} + n^{-s-1+\gamma}. \]

This concludes our proof of Lemma 22. \( \square \)

**Proof of Non-Uniform Dependence.** Consider the sequences of (actual) unique solutions to the Cauchy problem (4.5) to (4.7) given by \((u_{1,n}(x,t), \rho_{1,n}(x,t))\) and \((u_{-1,n}(x,t), \rho_{-1,n}(x,t))\) with initial data
\[ (u_{1,n}(x,0), \rho_{1,n}(x,0)) \text{ and } (u_{-1,n}(x,0), \rho_{-1,n}(x,0)) \]

respectively. Applying Lemma 22 we have
\[ \| (u_{\omega,n}(t) - u_{\omega,n}(t), \rho_{\omega,n}(t) - \rho_{\omega,n}(t)) \|_\gamma \lesssim n^{-\varepsilon_s}, \ 0 \leq t \leq T. \]
Furthermore, from our solution size estimate, we have that
\[ \|(u_{\omega,n}(t), \rho_{\omega,n}(t))\|_k \lesssim \|(u_{\omega,n}(0), \rho_{\omega,n}(0))\|_k \lesssim n^{-1} + n^{k-s} + n^{k-s-1} \]
for \( k > s \).

This implies that
\[ \|u_{\omega,n}(t)\|_{H^k} \lesssim \|u_{\omega,n}(0)\|_{H^k} + \|\rho_{\omega,n}(0)\|_{H^{k-1}}, \]
\[ \|\rho_{\omega,n}(t)\|_{H^{k-1}} \lesssim \|u_{\omega,n}(0)\|_{H^k} + \|\rho_{\omega,n}(0)\|_{H^{k-1}}. \]

Thus, by utilizing the triangle inequality, we obtain
\[ \|u_{\omega,n}(t) - u_{\omega,n}(0)\|_{H^k} \lesssim n^{-1} + n^{k-s} + n^{k-s-1} \lesssim n^{k-s}, \]
\[ \|\rho_{\omega,n}(t) - \rho_{\omega,n}(0)\|_{H^{k-1}} \lesssim n^{-1} + n^{k-s} + n^{k-s-1} \lesssim n^{k-s}. \]

Now we wish to use the following interpolation lemma.

**Lemma 23.** Let \( f \in H^s \) and \( s_1 < s < s_2 \). Then
\[ \|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \|f\|_{H^{s_2}}^{\frac{s-s_1}{s_2-s_1}}. \]

Let \( v(t) = u_{\omega,n}(t) - u_{\omega,n}(t) \) and \( \theta(t) = \rho_{\omega,n}(t) - \rho_{\omega,n}(t) \). Interpolating between the \( H^\gamma \) norm and the \( H^k \) norm to get an \( H^s \) estimate for \( v(t) \) and between the \( H^{\gamma-1} \) norm and the \( H^{k-1} \) norm to get an \( H^{s-1} \) estimate for \( \theta(t) \) where \( k = [s] + 2 \) yields
the following estimate:

\[
\|v\|_{H^s} \leq \|v\|_{H^{k-s}} \|v\|_{H^{k-s}}^{\frac{k-s}{k-s}} \lesssim (n^{-\varepsilon})^\frac{k-s}{k-s} \approx n^{-\beta_s}.
\]

\[
\|\theta\|_{H^{s-1}} \leq \|\theta\|_{H^{k-1}} \|\theta\|_{H^{k-1}}^{\frac{k-s}{k-s}} \lesssim (n^{-\varepsilon})^\frac{k-s}{k-s} \approx n^{-\beta_s}.
\]

where

\[
\beta_s = \frac{k-s}{k-s}.\]

Note that when \( k = [s] + 2 \) we have that \( \beta_s \) is greater than zero. At time \( t = 0 \), we have

\[
\|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s} = \|2n^{-1}\|_{H^s} \approx n^{-1} \to 0 \quad \text{as} \quad n \to \infty
\]

\[
\|\rho_{1,n}(0) - \rho_{-1,n}(0)\|_{H^{s-1}} = \|2n^{-1}\|_{H^{s-1}} \approx n^{-1} \to 0 \quad \text{as} \quad n \to \infty.
\]

However, at time \( t > 0 \) we have

\[
\|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s} \geq \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s} - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s}
\]

\[
- \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s}. \quad (4.27)
\]

and

\[
\|\rho_{1,n}(t) - \rho_{-1,n}(t)\|_{H^{s-1}} \geq \|\rho^{1,n}(t) - \rho^{-1,n}(t)\|_{H^{s-1}} - \|\rho^{1,n}(t) - \rho_{1,n}(t)\|_{H^{s-1}}
\]

\[
- \|\rho^{-1,n}(t) - \rho_{-1,n}(t)\|_{H^{s-1}}. \quad (4.28)
\]
Adding (4.27) and (4.28) we have that
\[
\|(u_{1,n}(t) - u_{-1,n}(t), \rho_{1,n}(t) - \rho_{-1,n}(t))\|_s \geq \|u_{1,n}^{1,n}(t) - u_{-1,n}^{1,1}(t)\|_{H^s} - cn^{-\beta_s} + \|\rho_{1,n}^{1,n}(t) - \rho_{-1,n}^{1,1}(t)\|_{H^{s-1}} - an^{-\beta_s}. 
\] (4.29)

Taking the limit infimum of both sides gives us
\[
\lim_{n \to \infty} \inf (\|(u_{1,n}(t) - u_{-1,n}(t), \rho_{1,n}(t) - \rho_{-1,n}(t))\|_s) 
\geq \lim_{n \to \infty} \inf (\|(u_{1,n}^{1,n}(t) - u_{-1,n}^{1,1}(t), \rho_{1,n}^{1,n}(t) - \rho_{-1,n}^{1,1}(t))\|_s). 
\]

Hence, to finish the argument for Theorem 2, we need only find a lower bound for the difference of known approximate solutions:
\[
u_{1,n}^{1,n}(t) - u_{-1,n}^{1,1}(t) = \rho_{1,n}^{1,n}(t) - \rho_{-1,n}^{1,1}(t) = 2n^{-1} + n^{-s}[\cos(nx - t) - \cos(nx + t)].
\]

Using the identity \(\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}\) yields
\[
u_{1,n}^{1,n}(t) - u_{-1,n}^{1,1}(t) = \rho_{1,n}^{1,n}(t) - \rho_{-1,n}^{1,1}(t) = 2n^{-1} + 2n^{-s} \sin(nx) \sin(t).
\]

Therefore,
\[
\|(u_{1,n}^{1,n}(t) - u_{-1,n}^{1,1}(t), \rho_{1,n}^{1,n}(t) - \rho_{-1,n}^{1,1}(t))\|_s \geq |\sin(t)| - n^{-1} + n^{-1} |\sin(t)|.
\]
Taking the limit infimum of both sides yields

$$
\lim_{n \to \infty} \inf (\|(u^{1,n}(t) - u^{-1,n}(t), \rho^{1,n}(t) - \rho^{-1,n}(t))\|_s) \gtrsim |\sin(t)|,
$$

which concludes our proof of Theorem 2. □
Here, we can duplicate the proofs for uniqueness and continuity of the flow map as well as the preliminary estimates and lifespan without any difficulty. The trouble lies in the existence of a solution on the real line. Furthermore, we must change our Friedrichs mollifiers to be non-periodic and thus we fix $j \in \mathcal{S}(\mathbb{R})$ such that $\hat{j}(0) = 1$. Then we may define $j_\varepsilon(x) \doteq (1/\varepsilon)j(x/\varepsilon)$. Therefore we have our Friedrichs mollifier to be $J_\varepsilon$ such that

$$J_\varepsilon f \doteq j_\varepsilon \ast f, \quad \varepsilon \in (0,1].$$

Now, we may proceed in the proof of existence.

**Proposition 5.** (*Existence*) There exists a solution $(u, \rho) \in C([0,T]; H^s \times H^{s-1} )$ to the Cauchy problem for the 2-component CH system (2.1) satisfying the solution size estimate given in (2.3).

**Proof.** We begin by defining the interval $I = [0,T]$ in order to simplify notation. Our proof revolves around refining the convergence of the family $\{(u_\varepsilon, \rho_\varepsilon)\}$ several times by extracting subsequences $\{(u_{\varepsilon_\nu}, \rho_{\varepsilon_\nu})\}$. After each such extraction, it is assumed that the resulting subsequence is relabeled $\{(u_\varepsilon, \rho_\varepsilon)\}$. 

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Weak* convergence in $L^\infty(I; H^s \times H^{s-1}(\mathbb{R}))$. The family $\{(u_\varepsilon, \rho_\varepsilon)\}$ is bounded in the space $C(I; H^s \times H^{s-1}(\mathbb{R})) \subset L^\infty(I; H^s \times H^{s-1}(\mathbb{R}))$. If we observe that the dual of the space $L^1(I, H^s \times H^{s-1}(\mathbb{R}))$ is $L^\infty(I, H^s \times H^{s-1}(\mathbb{R}))$, then Alaoglu’s theorem tells us that $\{(u_\varepsilon, \rho_\varepsilon)\}$ will be precompact in $\tilde{B} \left(0, 2 \left(\|u_0\|_{H^s(\mathbb{R})} + \|\rho_0\|_{H^{s-1}(\mathbb{R})}\right)\) \subset L^\infty(I, H^s \times H^{s-1}(\mathbb{R}))$ with respect to the weak* topology. Therefore, we may find a subsequence $\{(u_{\varepsilon,\nu}, \rho_{\varepsilon,\nu})\}$ that converges to an element in $\tilde{B} \left(0, 2 \left(\|u_0\|_{H^s(\mathbb{R})} + \|\rho_0\|_{H^{s-1}(\mathbb{R})}\right)\)$ weakly*.

Relabeling $\{(u_{\varepsilon,\nu}, \rho_{\varepsilon,\nu})\}$ as $\{(u_\varepsilon, \rho_\varepsilon)\}$ we will refine this sequence to converge strongly to $(u, \rho)$ in the $C(I; H^{s-1} \times H^{s-2}(\mathbb{R}))$ topology. Also, throughout the rest of the proof for existence, we shall apply a cut-off function $\varphi \in \mathcal{S}(\mathbb{R})$ to each of our terms in order to guarantee convergence in particular topologies along the real line.

**Lemma 24.** There exists a subsequence $\{(\varphi u_{\varepsilon,\nu}, \varphi \rho_{\varepsilon,\nu})\}$ of $\{(\varphi u_\varepsilon, \varphi \rho_\varepsilon)\}$ that converges strongly to $(\varphi u, \varphi \rho)$ in the space $C(I; H^{s-1} \times H^{s-2}(\mathbb{R}))$.

**Proof.** We will prove that the family $\{(\varphi u_{\varepsilon,\nu}, \varphi \rho_{\varepsilon,\nu})\}_{\varepsilon \in (0,1]}$ satisfies the hypothesis of Ascoli’s Theorem. We begin with the equicontinuity condition. For $t_1, t_2 \in I$ we have by the Mean Value Theorem

$$\|\varphi u_\varepsilon(t_1) - \varphi u_\varepsilon(t_2)\|_{H^{s-1}} \leq \sup_{t \in I} \|\partial_t u_\varepsilon(t)\|_{H^{s-1}} |t_1 - t_2|$$

$$\|\varphi \rho_\varepsilon(t_1) - \varphi \rho_\varepsilon(t_2)\|_{H^{s-2}} \leq \sup_{t \in I} \|\partial_t \rho_\varepsilon(t)\|_{H^{s-2}} |t_1 - t_2|.$$ 

Now using the bounds (3.27) and (3.28) for $\|\partial_t u_\varepsilon(t)\|_{H^{s-1}}$ and $\|\partial_t \rho_\varepsilon(t)\|_{H^{s-2}}$ we obtain
the inequality

\[ \| (\varphi u_\varepsilon(t_1) - \varphi u_\varepsilon(t_2), \varphi \rho \varepsilon(t_1) - \varphi \rho \varepsilon(t_2)) \|_{s-1} \lesssim |t_1 - t_2|, \]

which implies that for each \( t \in I \), the set \( U(t) = \{ (\varphi u_\varepsilon(t), \varphi \rho \varepsilon(t)) \} \) is bounded in \( H^s \times H^{s-1}(\mathbb{R}) \). We also see that \( U(t) \subset H^s \times H^{s-1}(\mathbb{R}) \) is precompact in \( H^{s-1} \times H^{s-2}(\mathbb{R}) \) as a consequence of Rellich’s Theorem. □

Our next goal is to demonstrate that the sequence \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \to (\varphi u, \varphi \rho) \) converges strongly in the space \( C(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})) \) for \( \gamma \in (0, 1) \). To accomplish this task we first need an interpolation result.

**Lemma 25.** For \( \varepsilon \in (0, 1) \) and \( \gamma \in (0, 1) \) we have \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \in C^\gamma(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})) \). Furthermore, the \( C^\gamma(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})) \) norm of \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \) is bounded by

\[ \| (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \|_{C^\gamma(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R}))} \lesssim \| (u_0, \rho_0) \|_s + \| (u_0, \rho_0) \|_2^{2} \quad (5.1) \]

**Proof.** By definition we have

\[ \| \varphi u_\varepsilon \|_{C^\gamma(I; H^{s-\gamma})} \equiv \sup_{t \in I} \| \varphi u_\varepsilon(t) \|_{H^{s-\gamma}} + \sup_{t \neq t'} \frac{\| \varphi u_\varepsilon(t) - \varphi u_\varepsilon(t') \|_{H^{s-\gamma}}}{|t - t'|^{\gamma}} \quad (5.2) \]

\[ \| \varphi \rho_\varepsilon \|_{C^\gamma(I; H^{s-\gamma-1})} \equiv \sup_{t \in I} \| \varphi \rho_\varepsilon(t) \|_{H^{s-\gamma-1}} + \sup_{t \neq t'} \frac{\| \varphi \rho_\varepsilon(t) - \varphi \rho_\varepsilon(t') \|_{H^{s-\gamma-1}}}{|t - t'|^{\gamma}}. \quad (5.3) \]
First, we will note that
\[
\sup_{t \in I} \| \varphi u_\varepsilon \|_{H^{s-\gamma}} \leq \| u_\varepsilon \|_{H^s} \quad \text{and} \quad \sup_{t \in I} \| \varphi \rho_\epsilon \|_{H^{s-\gamma-1}} \leq \| \rho_\varepsilon \|_{H^{s-1}}.
\]
Using our bound from (2.3) we have
\[
\sup_{t \in I} \| \varphi u_\varepsilon \|_{H^{s-\gamma}} + \sup_{t \in I} \| \varphi \rho_\epsilon \|_{H^{s-\gamma-1}} \leq 2 \left( \| u_0 \|_{H^s(T)} + \| \rho_0 \|_{H^{s-1}(T)} \right). \quad (5.4)
\]
For the right term of (5.2), using the inequality \( x^\gamma \leq 1 + x, \ x > 0 \), and the fact that
\[
\| \varphi u_\varepsilon(t) - \varphi u_\varepsilon(t') \|_{H^{s-\gamma}} \lesssim \| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}}
\]
we have
\[
\sup_{t \neq t'} \frac{\| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}}^2}{|t - t'|^{2\gamma}} = \sup_{t \neq t'} \int_{\mathbb{R}} \frac{(1 + k^2)^{s-\gamma} |\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(t')|^2}{|t - t'|^{2\gamma}} \leq \sup_{t \neq t'} \int_{\mathbb{R}} \frac{(1 + k^2)^{s} |\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(t')|^2}{|t - t'|^{2\gamma}} + \sup_{t \neq t'} \int_{\mathbb{R}} \frac{(1 + k^2)^{s-1} |\tilde{u}_\varepsilon(t) - \tilde{u}_\varepsilon(t')|^2}{|t - t'|^{2\gamma}} \lesssim \| u_\varepsilon \|_{C(I;H^{s})}^2 + \| \partial_t u_\varepsilon \|_{C(I;H^{s-1})}^2 \lesssim \| u_\varepsilon \|_{H^s}^2 + (\| u_\varepsilon \|_{H^s}^2 + \| \rho_\varepsilon \|_{H^{s-1}}^2)^2. \quad (5.5)
\]
Taking the square root of both sides of (5.5) we obtain
\[
\sup_{t \neq t'} \frac{\| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}}}{|t - t'|^{\gamma}} \lesssim \| u_\varepsilon \|_{H^s} + (\| u_\varepsilon \|_{H^s}^2 + \| \rho_\varepsilon \|_{H^{s-1}}^2)^2. \quad (5.6)
\]
Using the same analysis above, we obtain the following inequality for the right side.
of (5.3)

\[ \sup_{t \neq t'} \frac{\| \rho_\varepsilon(t) - \rho_\varepsilon(t') \|^2_{H^{s-\gamma-1}}}{{|t - t'|^{2\gamma}}} = \sup_{t \neq t'} \int_{\mathbb{R}} (1 + k^2)^{s-\gamma-1} \left| \hat{\rho}_\varepsilon(t) - \hat{\rho}_\varepsilon(t') \right|^2 \frac{dt}{|t - t'|^{2\gamma}} \]

\leq \sup_{t \neq t'} \int_{\mathbb{R}} (1 + k^2)^{s-1} \left| \hat{\rho}_\varepsilon(t) - \hat{\rho}_\varepsilon(t') \right|^2 \frac{dt}{|t - t'|^{2\gamma}} 

+ \sup_{t \neq t'} \int_{\mathbb{R}} (1 + k^2)^{s-2} \left| \hat{\rho}_\varepsilon(t) - \hat{\rho}_\varepsilon(t') \right|^2 \frac{dt}{|t - t'|^{2\gamma}} 

\lesssim \| \rho_\varepsilon \|^2_{C(I; H^{s-1})} + \| \partial_t \rho_\varepsilon \|^2_{C(I; H^{s-2})} 

\lesssim \| \rho_\varepsilon \|^2_{H^{s-1}} + \| u_\varepsilon \|^2_{H^s} \| \rho_\varepsilon \|^2_{H^{s-1}} \tag{5.7} \]

Taking the square root of both sides of (5.7) we obtain

\[ \sup_{t \neq t'} \frac{\| \rho_\varepsilon(t) - \rho_\varepsilon(t') \|_{H^{s-\gamma-1}}}{{|t - t'|^{\gamma}}} \lesssim \| \rho_\varepsilon \|_{H^{s-1}} + \| u_\varepsilon \|_{H^s} \| \rho_\varepsilon \|_{H^{s-1}}. \tag{5.8} \]

Combining (5.4), (5.6) and (5.8) gives us our desired bound. \( \square \)

**Lemma 26.** For \( \gamma \in (0, 1) \), there exists a subsequence \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \) of \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \) that converges strongly to \( (\varphi u, \varphi \rho) \) in the \( C(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})) \) topology. Furthermore, we have that \( (\varphi u_\varepsilon, \varphi \rho_\varepsilon) \rightarrow (\varphi u, \varphi \rho) \) in \( C(I, C^1(\mathbb{R}) \times C^1(\mathbb{R})) \).

**Proof.** We will once again use Ascoli's Theorem. The precompactness condition is established with the same argument as Lemma 8. For the condition of equicontinuity we observe have

\[ \| \varphi u_\varepsilon(t) - \varphi u_\varepsilon(t') \|_{H^{s-\gamma}(\mathbb{R})} \leq \| \varphi u_\varepsilon \|_{C^\gamma(I; H^{s-\gamma}(\mathbb{R}))} |t - t'|^{\gamma} \tag{5.9} \]

\[ \| \varphi \rho_\varepsilon(t) - \varphi \rho_\varepsilon(t') \|_{H^{s-\gamma-1}(\mathbb{R})} \leq \| \varphi \rho_\varepsilon \|_{C^\gamma(I; H^{s-\gamma-1}(\mathbb{R}))} |t - t'|^{\gamma}. \tag{5.10} \]
Adding (5.9) and (5.10) together and using the bound (5.1) we find that
\[ \| (\varphi u_\varepsilon(t) - \varphi u_\varepsilon(t'), \varphi \rho_\varepsilon(t) - \varphi \rho_\varepsilon(t')) \|_{s-\gamma} \lesssim |t - t'|^\gamma. \] (5.11)

Therefore, \((\varphi u_\varepsilon, \varphi \rho_\varepsilon)\) of \((\varphi u_\varepsilon, \varphi \rho_\varepsilon)\) converges strongly to \((\varphi u, \varphi \rho)\) in the \(C(I; H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R}))\) topology.

**Equicontinuity of \(\{(\varphi u_\varepsilon, \varphi \rho_\varepsilon)\}\) in \(C(I, C^1(\mathbb{R}) \times C^1(\mathbb{R}))\).** Choosing \(\gamma > 0\) small enough so that \(s - \gamma > 5/2\) we can apply the Sobolev Embedding Theorem to obtain
\[ \| \varphi u_\varepsilon(t) - \varphi u_\varepsilon(t') \|_{C^1} \lesssim \| u_\varepsilon(t) - u_\varepsilon(t') \|_{H^{s-\gamma}} \] (5.12)
\[ \| \varphi \rho_\varepsilon(t) - \varphi \rho_\varepsilon(t') \|_{C^1} \lesssim \| \rho_\varepsilon(t) - \rho_\varepsilon(t') \|_{H^{s-\gamma-1}}. \] (5.13)

Adding (5.12) and (5.13) and using (5.11) we have
\[ \| \varphi u_\varepsilon(t) - \varphi u_\varepsilon(t') \|_{C^1} + \| \varphi \rho_\varepsilon(t) - \varphi \rho_\varepsilon(t') \|_{C^1} \lesssim |t - t'|^\gamma, \]
which shows that the family \(\{(\varphi u_\varepsilon, \varphi \rho_\varepsilon)\}\) is equicontinuous is \(C(I, C^1 \times C^1)\).

**Compactness of \(\{(u_\varepsilon, \rho_\varepsilon)\}\) in \(C^1(\mathbb{R}) \times C^1(\mathbb{R})\).** Using estimate (2.3) gives
\[ \|(\varphi u_\varepsilon(t), \varphi \rho_\varepsilon(t))\|_{s-\gamma} \leq \|(\varphi u_\varepsilon(t), \varphi \rho_\varepsilon(t))\|_s \leq 2 \left( \| (u_0, \rho_0) \|_s \right), \]
which is finite. Therefore, by Rellich’s Lemma for each \(t \in I\) the set \(\{(\varphi u_\varepsilon, \varphi \rho_\varepsilon)\}\) is precompact in \(H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})\). That is, any subset of \(\{(\varphi u_\varepsilon, \varphi \rho_\varepsilon)\}\) contains a sequence \(\{(\varphi u_{\varepsilon_n}, \varphi \rho_{\varepsilon_n})\}\) which converges to an element \((\varphi u, \varphi \rho) \in H^{s-\gamma} \times H^{s-\gamma-1}(\mathbb{R})\).
Choosing $\gamma > 0$ small enough so that $s - \gamma > 5/2$ and applying the Sobolev Embedding Theorem gives

$$\|\varphi u_{\varepsilon_n}(t) - \varphi u_{\varepsilon}(t)\|_{C^1} + \|\varphi \rho_{\varepsilon_n}(t) - \varphi \rho_{\varepsilon}(t)\|_{C^1} \to 0 \text{ as } n \to \infty,$$

which shows that for each $t \in I$ the family $\{(\varphi u_{\varepsilon}, \varphi \rho_{\varepsilon})\}$ is precompact in $C^1(\mathbb{R}) \times C^1(\mathbb{R})$. Therefore, $(\varphi u_{\varepsilon}, \varphi \rho_{\varepsilon}) \to (\varphi u, \varphi \rho)$ in $C(I, C^1(\mathbb{R}) \times C^1(\mathbb{R}))$.

Verifying that the limit $(u, \rho)$ solves the 2-component CH system. As in the proof for the case on the torus, we will utilize Lemma 26 and Lemma 26.

**Lemma 27.** The sequence $\{(\partial_t \varphi u_{\varepsilon}, \partial_t \varphi \rho_{\varepsilon})\} \to (\partial_t \varphi u, \partial_t \varphi \rho)$ in $C(I, C(\mathbb{R}) \times C(\mathbb{R}))$.

Starting from the mollified i.v.p. (2.20) we have

$$\begin{align*}
\partial_t \varphi u_{\varepsilon} &= -\varphi J_{\varepsilon} [J_{\varepsilon} u_{\varepsilon}, \partial_x J_{\varepsilon} u_{\varepsilon}] - \varphi D^{-2} \partial_x (u_{\varepsilon}^2) - \frac{1}{2} \varphi D^{-2} \partial_x [(\partial_x u_{\varepsilon})^2] - \frac{\gamma}{2} \varphi D^{-2} \partial_x (\rho_{\varepsilon}^2), \\
\partial_t \varphi \rho_{\varepsilon} &= -\varphi J_{\varepsilon} [J_{\varepsilon} u_{\varepsilon}, \partial_x J_{\varepsilon} \rho_{\varepsilon}] - \varphi J_{\varepsilon} [J_{\varepsilon} \rho_{\varepsilon}, \partial_x J_{\varepsilon} u_{\varepsilon}].
\end{align*}$$

By the continuity of the operator $D^{-2} \partial_x$, we may immediately deduce the convergence of the nonlocal terms of (5.14) as

$$\begin{align*}
\varphi D^{-2} \partial_x (u_{\varepsilon}^2) &\to \varphi D^{-2} \partial_x (u^2) \\
\frac{1}{2} \varphi D^{-2} \partial_x [(\partial_x u_{\varepsilon})^2] &\to \frac{1}{2} \varphi D^{-2} \partial_x [(\partial_x u)^2] \\
\frac{\gamma}{2} \varphi D^{-2} \partial_x (\rho_{\varepsilon}^2) &\to \frac{\gamma}{2} \varphi D^{-2} \partial_x (\rho^2).
\end{align*}$$

To handle the mollified Burgers term of the first component of (5.14), we will first
prove that \( \varphi J_{\varepsilon u} \to \varphi u \) in \( C(I, C(\mathbb{R})) \). We have

\[
\| \varphi J_{\varepsilon u} - \varphi u \|_{C(I, C(\mathbb{R}))} \leq \| \varphi J_{\varepsilon u} - \varphi u_{\varepsilon v} \|_{C(I, C)} + \| \varphi u_{\varepsilon v} - \varphi u \|_{C(I, C(\mathbb{R}))}. \tag{5.15}
\]

For the first term of (5.15), choose \( 1/2 < r < s \). Then by Lemma 11, for \( t \in I \) we have

\[
\| \varphi J_{\varepsilon u} - \varphi u_{\varepsilon v} \|_{C(I, C(\mathbb{R}))} \lesssim \| \varphi J_{\varepsilon u} - \varphi u_{\varepsilon v} \|_{H^r(\mathbb{R})} \lesssim \| I - J_{\varepsilon v} \|_{C(\mathbb{R})} \| u_{\varepsilon v} \|_{H^r(\mathbb{R})} \| \varphi \|_{H^r(\mathbb{R})} \lesssim \| I - J_{\varepsilon v} \|_{C(\mathbb{R})} \| u_{\varepsilon v} \|_{H^s(\mathbb{R})} = o(\varepsilon^{s-r}).
\]

For the second term of (5.15), we observe that Lemma 26 and the Sobolev Embedding Theorem imply that \( \| \varphi u_{\varepsilon v} - \varphi u \|_{C(I, C(\mathbb{R}))} \to 0 \) and that \( \varphi u \in C(I, C(\mathbb{R})) \). Now we examine \( \varphi \partial_x u \) as above to obtain

\[
\| \varphi J_{\varepsilon v} \partial_x u_{\varepsilon v} - \varphi \partial_x u \|_{C(I, C(\mathbb{R}))} \leq \| \varphi J_{\varepsilon v} \partial_x u_{\varepsilon v} - \varphi \partial_x u_{\varepsilon v} \|_{C(I, C(\mathbb{R}))} + \| \varphi \partial_x u_{\varepsilon v} - \varphi \partial_x u \|_{C(I, C(\mathbb{R}))}. \tag{5.16}
\]

For the first term of (5.16), choose \( 1/2 < r < s - 1 \). Then by Lemma 11 for \( t \in I \)
we have
\[
\|\phi J_{\epsilon\nu} \partial_x u_{\epsilon\nu} - \phi \partial_x u_{\epsilon\nu}\|_{C(I, C(R))} \lesssim \|\phi J_{\epsilon\nu} \partial_x u_{\epsilon\nu} - \phi \partial_x u_{\epsilon\nu}\|_{H^r(R)}
\]
\[
\lesssim \|I - J_{\epsilon\nu}\|_{L(H^{s-1}, H^r(R))} \|\partial_x u_{\epsilon\nu}\|_{H^{s-1}(R)} \|\phi\|_{H^r(R)}
\]
\[
\lesssim \|I - J_{\epsilon\nu}\|_{L(H^{s-1}, H^r(R))} \|\partial_x u_{\epsilon\nu}\|_{H^{s-1}(R)}
\]
\[
= o(\epsilon^{s-r-1}).
\]

For the second term of (5.16), we observe that \(\|\phi u_{\epsilon\nu} - \phi u\|_{C(I,C(R))} \to 0\) implies \(\|\phi \partial_x u_{\epsilon\nu} - \phi \partial_x u\|_{C(I,C(R))} \to 0\). Similarly, this can be shown for the mollified terms of the second component of (5.14).

Thus, proceeding via additive and multiplicative properties of limits we may conclude that \(\{(\partial_t \phi u_{\epsilon\nu}, \partial_t \phi \rho_{\epsilon\nu})\} \to (\partial_t \phi u, \partial_t \phi \rho)\) in \(C(I, C(R) \times C(R))\). \(\square\)

**Proposition 6.** The solution \((u, \rho)\) to the 2-component CH system i.v.p. is an element of the space \(C(I; H^s \times H^{s-1})\).

Here we can duplicate the proof from Proposition 2 on the torus. Thus, we have shown existence of solutions on the real line. Furthermore, since we can duplicate the proofs for uniqueness and continuous dependence of the data-to-solution map from the case of the torus, we have shown that the 2-Component Camassa-Holm system is well-posed on the real line.
CHAPTER 6

NON-UNIFORM DEPENDENCE ON THE REAL LINE

We recall that we have already proven non-uniform dependence for the periodic case. Now we shall extend the argument to the non-periodic case for our 2-component CH system.

Construction of approximate solutions. Here we shall construct a two-parameter family of approximate solutions \( u^{\omega,n} = u^{\omega,n}(x,t) \) and \( \rho^{\omega,n} = \rho^{\omega,n}(x,t) \) in which each member consists of two parts. Thus, we have the following for \( u^{\omega,n} \) and \( \rho^{\omega,n} \):

\[
u^{\omega,n} = u_t + u^h \quad \rho^{\omega,n} = \rho_t + \rho^h.
\]

The high frequency parts \( u^h \) and \( \rho^h \) are given by

\[
u^h = u^{h,\omega,n}(x,t) = n^{-\delta/2-s} \varphi\left(\frac{x}{n^\delta}\right) \cos(nx - \omega t)
\]

\[
\rho^h = \rho^{h,\omega,n}(x,t) = n^{-\delta/2-s} \psi\left(\frac{x}{n^\delta}\right) \cos(nx - \omega t)
\]

and \((u^h, \rho^h)\) is not a solution of the 2-component CH system. Here we have \( \varphi, \psi \in C^\infty \)
such that

\[ \varphi(x) = \psi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ 0, & \text{if } |x| \geq 2. \end{cases} \]

The low frequency part \((u_l, \rho_l)\) is a solution to the 2-component CH system and the initial value \((u_l(x, 0), \rho_l(x, 0))\) is as follows:

\[
\begin{align*}
u_l(x, 0) &= \omega n^{-1} \tilde{\varphi}(\frac{x}{n^\delta}) \\
\rho_l(x, 0) &= \omega n^{-1} \tilde{\psi}(\frac{x}{n^\delta}), \quad x, t \in \mathbb{R}.
\end{align*}
\]

where \(\tilde{\varphi}, \tilde{\psi} \in C_0^\infty(\mathbb{R})\) such that

\[ \tilde{\varphi}(x) = 1, \quad \text{if } x \in \text{supp } \varphi \cup \text{supp } \psi. \]

Before we proceed to the error estimates, we require the following lemmas.

**Lemma 28.** Let \(\psi \in \mathcal{S}(\mathbb{R}), \delta > 0\) and \(\alpha \in \mathbb{R}\). Then for any \(s \geq 0\) we have that

\[
\lim_{n \to \infty} n^{-\frac{1}{2}\delta - s} \|\psi(\frac{x}{n^\delta}) \cos(nx - \omega t)\|_{H^s(\mathbb{R})} = \frac{1}{\sqrt{2}} \|\psi\|_{L^2(\mathbb{R})}. \tag{6.1}
\]

Relation \([6.1]\) is also true if cosine is replaced by sine.

**Proof.** Since

\[
\left(\psi(\frac{x}{n^\delta}) \cos(nx - \omega t)\right) (\xi) = \frac{1}{2} n^\delta [e^{-i\alpha} \hat{\psi}(n^\delta (\xi - n)) + e^{i\alpha} \hat{\psi}(n^\delta (\xi + n))],
\]

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we have that

\[
n^{-\delta-2s}\|\psi(n^\delta)\cos(nx-\alpha)\|^2_{H^s(\mathbb{R})} = \\
= \frac{n^{-2s+\delta}}{8\pi} \int_\mathbb{R} (1 + \xi^2)^s |\hat{\psi}(n^\delta(\xi - n)) + e^{i\alpha} \hat{\psi}(n^\delta(\xi + n))|^2 d\xi \\
= \frac{n^{-2s+\delta}}{8\pi} \left[ \int_\mathbb{R} (1 + \xi^2)^s |\hat{\psi}(n^\delta(\xi - n))|^2 d\xi \\
+ \int_\mathbb{R} (1 + \xi^2)^s |\hat{\psi}(n^\delta(\xi + n))|^2 d\xi \\
+ 2 \int_\mathbb{R} (1 + \xi^2)^s \Re\left[ e^{-2i\alpha} \hat{\psi}(n^\delta(\xi - n))\hat{\psi}(n^\delta(\xi - n)) \right] d\xi \right].
\]

In the first and third integrals we make the change of variables \( \eta = n^\delta(\xi - n) \), while in the second we let \( \eta = n^\delta(\xi + n) \). Thus, we have

\[
n^{-\delta-2s}\|\psi(n^\delta)\cos(nx-\alpha)\|^2_{H^s(\mathbb{R})} = \\
= \frac{n^{-2s}}{8\pi} \left[ \int_\mathbb{R} \left( 1 + \left( \frac{\eta}{n^\delta} + n \right)^2 \right)^s |\hat{\psi}(\eta)|^2 d\eta \\
+ \int_\mathbb{R} \left( 1 + \left( \frac{\eta}{n^\delta} - n \right)^2 \right)^s |\hat{\psi}(\eta)|^2 d\eta \\
+ 2 \int_\mathbb{R} \left( 1 + \left( \frac{\eta}{n^\delta} + n \right)^2 \right)^s \Re\left[ e^{-2i\alpha} \hat{\psi}(n\eta)\hat{\psi}(\eta + 2n^{\delta+1}) \right] d\eta \right].
\]
Moving the factor $n^{-2s}$ inside the integral gives

$$n^{-\delta-2s} \| \psi \left( \frac{x}{n^\delta} \right) \cos(nx - \alpha) \|^2_{H^s(\mathbb{R})} =$$

$$= \frac{1}{8\pi} \left[ \int_{\mathbb{R}} \left( \frac{1}{n^2} + \left( \frac{\eta}{n^{\delta+1}} + 1 \right)^2 \right)^s |\hat{\psi}(\eta)|^2 d\etaight]
+ \int_{\mathbb{R}} \left( \frac{1}{n^2} + \left( \frac{\eta}{n^{\delta+1}} - 1 \right)^2 \right)^s |\hat{\psi}(n)|^2 d\eta
+ 2 \int_{\mathbb{R}} \left( \frac{1}{n^2} + \left( \frac{\eta}{n^{\delta+1}} + 1 \right)^2 \right)^s \Re\{e^{-2i\alpha} \hat{\psi}(n) \hat{\psi}(\eta + 2n^{\delta+1})\} d\xi].$$

Since $\psi \in S(\mathbb{R})$ we have that $\hat{\psi}(\eta + 2n^{\delta+1}) \to 0$ as $n \to \infty$. Therefore, by applying the Dominated Convergence Theorem we see that the third integral goes to zero while the other two go to $\| \hat{\psi} \|^2_{L^2(\mathbb{R})}$. Therefore, we have that

$$\lim_{n \to \infty} n^{-\delta-2s} \| \psi \left( \frac{x}{n^\delta} \right) \cos(nx - \alpha) \|^2_{H^s(\mathbb{R})} = \frac{1}{4\pi} \| \hat{\psi} \|^2_{L^2(\mathbb{R})} = \frac{1}{2} \| \psi \|^2_{L^2(\mathbb{R})},$$

which proves the lemma. □

**Lemma 29.** Let $s \geq 0$. For any $\psi \in S(\mathbb{R})$ we have

$$\| \psi \left( \frac{x}{n^\delta} \right) \|_{H^s(\mathbb{R})} \leq n^{\delta/2} \| \psi \|_{H^s(\mathbb{R})}. \tag{6.2}$$

**Proof.** Using the relation $\widehat{\psi(x/\lambda)}(\xi) = \lambda \widehat{\psi}(\lambda \xi)$ and making the change of variables
η = n^δ ξ we obtain

$$\|\psi \left( \frac{x}{n^\delta} \right) \|_{H^s(\mathbb{R})}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} \left( 1 + \xi^2 \right)^s n^\delta \hat{\psi}(n^\delta \xi) d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \left( 1 + \frac{\eta^2}{n^{2\delta}} \right)^s \cdot n^{2\delta} \left| \hat{\psi}(\eta) \right|^2 d\eta$$

$$= n^\delta \cdot \frac{1}{2\pi} \int_{\mathbb{R}} \left( 1 + \frac{\eta^2}{n^{2\delta}} \right)^s \left| \hat{\psi}(\eta) \right|^2 d\eta$$

$$\leq n^\delta \cdot \frac{1}{2\pi} \int_{\mathbb{R}} \left( 1 + \eta^2 \right)^s \left| \hat{\psi}(\eta) \right|^2 d\eta$$

$$= n^\delta \left\| \psi \right\|_{H^s(\mathbb{R})}^2.$$

By taking the squareroot of both sides of the above inequality, we achieve (6.2) which proves Lemma 29.

**Lemma 30.** Let ω be bounded, 0 < δ < 2 and n >> 1. Then the initial value problem (2.1) where (u, ρ) is now replaced by (u_l, ρ_l) has a unique smooth solution (u_l, ρ_l) ∈ C([0, 1]; H^s × H^{s-1}(\mathbb{R})), for all s > 5/2 and satisfying the estimate

$$\|(u_l, \rho_l)\|_s \leq c_s n^{-1+\delta/2}.$$  

**Proof.** Using inequality (6.2) we have that the initial data u_l(0) and ρ_l(0) satisfy the estimate

$$\|u_l(0)\|_{H^s(\mathbb{R})} \leq |\omega| n^{-1+\delta/2} \|\tilde{\psi}\|_{H^s(\mathbb{R})},$$

$$\|\rho_l(0)\|_{H^{s-1}(\mathbb{R})} \leq |\omega| n^{-1+\delta/2} \|\tilde{\psi}\|_{H^{s-1}(\mathbb{R})}.$$
which for $\omega$ bounded decays if $\delta < 2$.

Next, using the estimate (2.3) and Theorem 1, we have that the lifespan $T$ of the solution $(u_l(t), \rho_l(t))$ satisfies

$$T \geq \frac{1}{2c_s \| (u_l(0), \rho_l(0)) \|_s} \geq \frac{c_s'}{n^{-1+\delta/2}} \geq 1,$$

for $n \gg 1$, since $\delta < 2$. Finally, if $s \geq 0$ then from estimate (2.3) we have

$$\|(u_l(t), \rho_l(t))\|_s \leq \|(u_l(t), \rho_l(t))\|_{s+2} \leq c_s \|(u_l(0), \rho_l(0))\|_{s+2} \leq c_s n^{-1+\delta/2}. \quad \square$$

Now, substituting the approximate solution $(u^{\omega,n}, \rho^{\omega,n})$ into the 2-component CH system and taking into consideration that $(u_l, \rho_l)$ is a solution we obtain the following errors

$$E = \partial_t u_h + u_l \partial_x u_h + u^h \partial_x u_l + u^h \partial_x u^h$$

$$+ D^{-2} \partial_x [2u_l u^h + (u^h)^2 + \partial_x u_l \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2 + \sigma \rho_l \rho^h + \frac{\sigma}{2} (\rho^h)^2]$$

$$F = \partial_t \rho_h + u_l \partial_x \rho_h + u^h \partial_x \rho_l + \rho_l \partial_x u^h + \rho^h \partial_x u_l + \rho^h \partial_x u^h + u^h \partial_x \rho^h.$$ 

Next we wish to compute our time and spacial derivatives of our high frequency terms. Computing $\partial_t u^h$ and $\partial_t \rho^h$ gives us

$$\partial_t u^h = \omega n^{-\delta/2-s} \varphi \left( \frac{x}{n^s} \right) \sin(nx - \omega t)$$

$$\partial_t \rho^h = \omega n^{-\delta/2-s} \psi \left( \frac{x}{n^s} \right) \sin(nx - \omega t).$$
Since \( \tilde{\varphi} = 1 \) on supp \( \varphi \cup \text{supp } \psi \), we have the following result

\[
\partial_t u^h = \omega \tilde{\varphi}(\frac{x}{n^\delta}) n^{-\delta/2-s} \varphi(\frac{x}{n^\delta}) \sin(nx - \omega t) = nu_l(x, 0) n^{-\delta/2-s} \varphi(\frac{x}{n^\delta}) \sin(nx - \omega t),
\]

(6.3)

\[
\partial_t \rho^h = \omega \tilde{\varphi}(\frac{x}{n^\delta}) n^{-\delta/2-s} \psi(\frac{x}{n^\delta}) \sin(nx - \omega t) = nu_l(x, 0) n^{-\delta/2-s} \psi(\frac{x}{n^\delta}) \sin(nx - \omega t).
\]

(6.4)

Computing the spacial derivatives of our high frequency terms yields

\[
\partial_x u^h(x, t) = -n \cdot n^{-\delta/2-s} \varphi(\frac{x}{n^\delta}) \sin(nx - \omega t) + n^{-\frac{3}{2}\delta-s} \partial_x \varphi(\frac{x}{n^\delta}) \cos(nx - \omega t),
\]

(6.5)

\[
\partial_x \rho^h(x, t) = -n \cdot n^{-\delta/2-s} \psi(\frac{x}{n^\delta}) \sin(nx - \omega t) + n^{-\frac{3}{2}\delta-s} \partial_x \psi(\frac{x}{n^\delta}) \cos(nx - \omega t).
\]

(6.6)

Using (6.3)-(6.6) we have that

\[
\partial_t u^h + u_l \partial_x u^h = n [u_l(x, 0) - u_l(x, t)] n^{-\delta/2-s} \varphi(\frac{x}{n^\delta}) \sin(nx - \omega t)
+ u_l(x, t) n^{-\frac{3}{2}\delta-s} \partial_x \varphi(\frac{x}{n^\delta}) \cos(nx - \omega t),
\]

\[
\partial_t \rho^h + u_l \partial_x \rho^h = n [u_l(x, 0) - u_l(x, t)] n^{-\delta/2-s} \psi(\frac{x}{n^\delta}) \sin(nx - \omega t)
+ u_l(x, t) n^{-\frac{3}{2}\delta-s} \partial_x \psi(\frac{x}{n^\delta}) \cos(nx - \omega t).
\]

Therefore, we have that the following errors \( E \) and \( F \) of the approximate solution

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\((u^{\omega,n}, \rho^{\omega,n})\) are given by

\[
E = E_1 + E_2 + \cdots + E_6 \\
F = F_1 + F_2 + \cdots + F_7
\]

where

\[
E_1 = n [u_t(x, 0) - u_t(x, t)] n^{-s/2-s} \varphi\left(\frac{x}{n^s}\right) \sin(nx - \omega t) \\
E_2 = u_t(x, t) n^{-s/2-s} \partial_x \varphi\left(\frac{x}{n^s}\right) \cos(nx - \omega t) \\
E_3 = u^h \partial_x u_t \\
E_4 = u^h \partial_x u^h \\
E_5 = D^{-2} \partial_x [2u_t u^h + (u^h)^2 + \partial_x u_t \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2] \\
E_6 = D^{-2} \partial_x [\sigma \rho_t \rho^h + \frac{\sigma}{2} (\rho^h)^2]
\]
and

\[ \begin{align*}
F_1 &= n \left[ u_l(x, 0) - u_l(x, t) \right] n^{-\delta/2-s} \psi \left( \frac{x}{n^\delta} \right) \sin(nx - \omega t) \\
F_2 &= u_l(x, t) n^{-\frac{3}{2}\delta-s} \partial_x \psi \left( \frac{x}{n^\delta} \right) \cos(nx - \omega t) \\
F_3 &= u_h \partial_x \rho_l \\
F_4 &= \rho_l \partial_x u_h \\
F_5 &= \rho_h \partial_x u_l \\
F_6 &= \rho_h \partial_x u_h \\
F_7 &= u_h \partial_x \rho_h.
\end{align*} \]

Using the values of \( \gamma \) from the periodic case, we will proceed to take the \( H^\gamma \) norm of each error \( E_j \) for \( j = 1, \ldots, 5 \).

**Estimating the \( H^\gamma \) norm of \( E_1 \).** We have

\[
\| E_1 \|_{H^\gamma(\mathbb{R})} = n^{1-\delta/2-s} \| \varphi \left( \frac{x}{n^\delta} \right) \sin(nx - \omega t) [u_l(x, 0) - u_l(x, t)] \|_{H^\gamma(\mathbb{R})}.
\]

Since we have that \( \gamma > 1/2 \) we can use the algebra property which yields

\[
\| E_1 \|_{H^\gamma(\mathbb{R})} \lesssim n^{1-\delta/2-s} \| \varphi \left( \frac{x}{n^\delta} \right) \sin(nx - \omega t) \|_{H^\gamma(\mathbb{R})} \| u_l(x, 0) - u_l(x, t) \|_{H^\gamma(\mathbb{R})}.
\]  \hspace{1cm} (6.7)

We know that

\[
\| \varphi \left( \frac{x}{n^\delta} \right) \sin(nx - \omega t) \|_{H^\gamma(\mathbb{R})} \lesssim n^{\delta/2+\gamma}.
\]  \hspace{1cm} (6.8)

Thus, we concern ourselves with the \( H^\gamma \) norm of the difference \( u_l(x, t) - u_l(x, 0) \).
Applying the Fundamental Theorem of Calculus in the time variable gives us

\[ u_t(x, t) - u_t(x, 0) = \int_0^t \partial_t u_t(x, \tau) d\tau. \]  \hspace{1cm} (6.9)

Taking the \( H^\gamma \) norm of the space variable to both sides of (6.9) and passing the norm through the integral gives

\[ \| u_t(x, t) - u_t(x, 0) \|_{H^\gamma(\mathbb{R})} \leq \int_0^t \| \partial_t u_t(x, \tau) \|_{H^\gamma(\mathbb{R})} d\tau. \]  \hspace{1cm} (6.10)

Now we have that

\[
\| \partial_t u_t(x, \tau) \|_{H^\gamma(\mathbb{R})} \leq \| u_t \|_{H^\gamma(\mathbb{R})} \| \partial_x u_t \|_{H^\gamma(\mathbb{R})} + \| D^{-2} \partial_x [(u_t)^2 + \frac{1}{2} (\partial_x u_t)^2 + \frac{b}{2} (\rho_t)^2] \|_{H^\gamma(\mathbb{R})}
\]
\[
\lesssim \| u_t \|_{H^{\gamma+1}(\mathbb{R})} \| \partial_x u_t \|_{H^\gamma(\mathbb{R})} + \| (u_t)^2 \|_{H^\gamma(\mathbb{R})} + \| (\partial_x u_t)^2 \|_{H^{\gamma-1}(\mathbb{R})} + \| (\rho_t)^2 \|_{H^\gamma(\mathbb{R})}
\]
\[
\lesssim \| u_t \|_{H^1(\mathbb{R})} \| u_t \|_{H^2(\mathbb{R})} + \| (u_t)^2 \|_{H^\gamma(\mathbb{R})} + \| (\partial_x u_t)^2 \|_{H^{\gamma-1}(\mathbb{R})} + \| (\rho_t)^2 \|_{H^\gamma(\mathbb{R})}
\]
\[
\lesssim \| u_t \|_{H^2(\mathbb{R})}^2 + \| u_t \|_{H^\gamma(\mathbb{R})}^2 + \| \partial_x u_t \|_{H^\gamma(\mathbb{R})}^2 + \| \rho_t \|_{H^\gamma(\mathbb{R})}^2
\]
\[
\lesssim \| u_t \|_{H^2(\mathbb{R})}^2 + \| \rho_t \|_{H^1(\mathbb{R})}.
\]

From Lemma 30 we see that by the last inequality

\[ \| \partial_t u_t(x, t) \|_{H^\gamma(\mathbb{R})} \lesssim n^{-2+\delta}. \]  \hspace{1cm} (6.11)

Substituting (6.11) back into (6.10) we obtain

\[ \| u_t(x, 0) - u_t(x, t) \|_{H^\gamma(\mathbb{R})} \lesssim n^{-2+\delta}. \]  \hspace{1cm} (6.12)
Finally, substituting (6.8) and (6.12) back into (6.7) we find that

$$\|E_1\|_{H^\gamma(\mathbb{R})} \lesssim n^{-s-1+\gamma+\delta}, \quad n \gg 1.$$ 

**Estimating the $H^\gamma$ norm of $E_2$.** From $E_2$ we obtain

$$\|E_2\|_{H^\gamma(\mathbb{R})} = \|u_t(x,t)n^{-\frac{3}{2}\delta-s}\partial_x \varphi\left(\frac{x}{n^\delta}\right)\cos(nx-\omega t)\|_{H^\gamma(\mathbb{R})}$$

$$\lesssim n^{-\frac{3}{2}\delta-s}\|\partial_x \varphi\left(\frac{x}{n^\delta}\right)\cos(nx-\omega t)\|_{H^\gamma(\mathbb{R})}\|u_t(x,t)\|_{H^\gamma(\mathbb{R})}$$

$$\lesssim n^{-\frac{3}{2}\delta-s} \cdot n^{\frac{1}{2}\delta+\gamma} \cdot n^{-1+\frac{1}{2}\delta}$$

$$\lesssim n^{-s-\delta/2-1+\gamma}.$$ 

**Estimating the $H^\gamma$ norm of $E_3$.** We have that

$$\|E_3\|_{H^\gamma(\mathbb{R})} = \|u^h \partial_x u\|_{H^\gamma(\mathbb{R})}$$

$$\lesssim \|u^h\|_{H^\gamma(\mathbb{R})}\|\partial_x u\|_{H^\gamma(\mathbb{R})}$$

$$\lesssim \|u^h\|_{H^\gamma(\mathbb{R})}\|u_t\|_{H^2(\mathbb{R})}$$

$$\lesssim n^{-s+\gamma} \cdot n^{-1+\delta/2}$$

$$\lesssim n^{-s-1+\delta/2+\gamma}.$$ 

**Estimating the $H^\gamma$ norm of $E_4$.** Using the refined algebra property

$$\|fg\|_{H^s} \leq C \left[\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{H^s}\right]$$

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we obtain
\[
\|E_4\|_{H^\gamma(\mathbb{R})} = \|u^h \partial_x u^h\|_{H^\gamma(\mathbb{R})} \\
\lesssim \|u^h\|_{H^\gamma(\mathbb{R})} \|\partial_x u^h\|_{L^\infty(\mathbb{R})} + \|u^h\|_{L^\infty(\mathbb{R})} \|\partial_x u^h\|_{H^{\gamma+1}(\mathbb{R})} \\
\lesssim n^{-s+\gamma} & \cdot n^{-s-\delta/2+1} + n^{-s-\delta/2} \cdot n^{-s+\gamma+1} \\
\lesssim n^{-2s-\delta/2+1+\gamma}.
\]

Estimating the $H^\gamma$ norm of $E_5$. Here we have
\[
\|E_5\|_{H^\gamma(\mathbb{R})} = \|D^{-2}\partial_x [2u_t u^h + (u^h)^2 + \partial_x u_t \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2]\|_{H^\gamma(\mathbb{R})} \\
\lesssim \|2u_t u^h + (u^h)^2 + \partial_x u_t \partial_x u^h + \frac{1}{2} (\partial_x u^h)^2\|_{H^{\gamma-1}(\mathbb{R})} \\
\lesssim \|2u_t u^h\|_{L^2(\mathbb{R})} + \|(u^h)^2\|_{L^2(\mathbb{R})} + \|\partial_x u_t \partial_x u^h\|_{L^2(\mathbb{R})} + \|(\partial_x u^h)^2\|_{L^2(\mathbb{R})} \\
\lesssim \|u^h\|_{L^\infty} \|u_t\|_{H^2} + \|u^h\|_{L^\infty} \|u^h\|_{L^2} + \|\partial_x u^h\|_{L^\infty} \|u_t\|_{H^2} + \|\partial_x u^h\|_{L^\infty} \|u^h\|_{H^1} \\
\lesssim n^{-\delta/2-s} \cdot n^{-1+\delta/2} + n^{-\delta/2-s} \cdot n^{-s} + n^{-\delta/2-s+1} \cdot n^{-1+\delta/2} + n^{-\delta/2-s+1} \cdot n^{-s+1} \\
\lesssim n^{-s-1} + n^{-2s-\delta/2} + n^{-s} + n^{-2s-\delta/2+2}.
\]
Estimating the $H^γ$ norm of $E_6$. Here we have

$$
\|E_6\|_{H^γ(\mathbb{R})} = \|D^{-2}\partial_x [\sigma \rho \rho^h + \frac{\sigma}{2}(\rho^h)^2]\|_{H^γ(\mathbb{R})}
\lesssim \|\sigma \rho \rho^h + \frac{\sigma}{2}(\rho^h)^2\|_{H^{γ-1}(\mathbb{R})}
\lesssim \|\rho \rho^h\|_{L^2(\mathbb{R})} + \|(\rho^h)^2\|_{L^2(\mathbb{R})}
\lesssim \|\rho^h\|_{L^∞(\mathbb{R})}\|\rho\|_{H^2(\mathbb{R})} + \|\rho^h\|_{L^∞}\|\rho^h\|_{L^2(\mathbb{R})}
\lesssim n^{-δ/2-s} \cdot n^{-1+δ/2} + n^{-δ/2-s} \cdot n^{-s}
\lesssim n^{-s-1} + n^{-2s-δ/2}.
$$

Now we wish to compute the $H^{γ-1}$ norm of each error $F_j$ for $j = 1, \ldots, 7$.

**Estimating the $H^{γ-1}$ norm of $F_1$.** Using estimates from the section on calculating the $H^γ$ norm of $E_1$ along with Lemma 3, we obtain

$$
\|F_1\|_{H^{γ-1}(\mathbb{R})} = n^{1-δ/2-s}\|\psi(\frac{x}{n^δ})\sin(nx - ωt)[u_l(x, 0) - u_l(x, t)]\|_{H^{γ-1}(\mathbb{R})}
\lesssim n^{1-δ/2-s}\|\psi(\frac{x}{n^δ})\sin(nx - ωt)\|_{H^{γ-1}(\mathbb{R})}\|u_l(x, 0) - u_l(x, t)\|_{H^γ(\mathbb{R})}
\lesssim n^{1-δ/2-s}\|\psi(\frac{x}{n^δ})\|_{H^γ(\mathbb{R})}\|\sin(nx - ωt)\|_{H^{γ-1}(\mathbb{R})}\|u_l(x, 0) - u_l(x, t)\|_{H^γ(\mathbb{R})}
\lesssim n^{-s+δ-2+γ}.
$$

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Estimating the $H^{\gamma-1}$ norm of $F_2$. Applying Lemma 3 again, we achieve

$$\| F_2 \|_{H^{\gamma-1}(\mathbb{R})} = n^{\frac{3}{2} \delta - s} \| u_t(x,t) \partial_x \psi(\frac{x}{n^\delta}) \cos(nx - \omega t) \|_{H^{\gamma-1}(\mathbb{R})}$$

$$\lesssim n^{\frac{3}{2} \delta - s} \| \partial_x \psi(\frac{x}{n^\delta}) \cos(nx - \omega t) \|_{H^{\gamma-1}(\mathbb{R})} \| u_t \|_{H^{\gamma}(\mathbb{R})}$$

$$\lesssim n^{\frac{3}{2} \delta - s} \| \partial_x \psi(\frac{x}{n^\delta}) \|_{H^{\gamma-1}(\mathbb{R})} \| \cos(nx - \omega t) \|_{H^{\gamma}(\mathbb{R})} \| u_t \|_{H^{\gamma}(\mathbb{R})}$$

$$\lesssim n^{\frac{-s-\delta}{2} - 1 + \gamma}.$$ 

Estimating the $H^{\gamma-1}$ norm of $F_3$. Here we have

$$\| F_3 \|_{H^{\gamma-1}(\mathbb{R})} = \| u^h \partial_x \rho_l \|_{H^{\gamma-1}(\mathbb{R})}$$

$$\lesssim \| u^h \|_{H^{\gamma}(\mathbb{R})} \| \partial_x \rho_l \|_{H^{\gamma-1}(\mathbb{R})}$$

$$\lesssim \| u^h \|_{H^{\gamma}(\mathbb{R})} \| \rho_l \|_{H^{\gamma}(\mathbb{R})}$$

$$\lesssim n^{-s+\gamma} \cdot n^{-1+\delta/2}$$

$$\lesssim n^{-s-1+\delta/2+\gamma}.$$ 

Estimating the $H^{\gamma-1}$ norm of $F_4$. Here we apply Lemma 3 again and obtain

$$\| F_4 \|_{H^{\gamma-1}(\mathbb{R})} = \| \rho_l \partial_x u^h \|_{H^{\gamma-1}(\mathbb{R})}$$

$$\lesssim \| \rho_l \|_{H^{\gamma}(\mathbb{R})} \| \partial_x u^h \|_{H^{\gamma-1}(\mathbb{R})}$$

$$\lesssim \| \rho_l \|_{H^{\gamma}(\mathbb{R})} \| u^h \|_{H^{\gamma}(\mathbb{R})}$$

$$\lesssim n^{-1+\delta/2} \cdot n^{-s+\gamma}$$

$$\lesssim n^{-s+\delta/2-1+\gamma}.$$
**Estimating the** $H^{\gamma-1}$ **norm of** $F_5$. Duplicating the process from the calculation of the $H^{\gamma-1}$ norm of $F_3$ we find that

$$\|F_5\|_{H^{\gamma_1}(\mathbb{R})} \lesssim n^{-s-1+\delta/2+\gamma}.$$ 

**Estimating the** $H^{\gamma-1}$ **norm of** $F_6$. Here we apply Lemma 3 and achieve

$$\|F_6\|_{H^{\gamma-1}(\mathbb{R})} = \|\rho_l \partial_x u^h\|_{H^{\gamma-1}(\mathbb{R})} \lesssim \|\rho_l\|_{H^{\gamma}(\mathbb{R})} \|\partial_x u^h\|_{H^{\gamma-1}(\mathbb{R})} \lesssim \|\rho_l\|_{H^{\gamma}(\mathbb{R})} \|u^h\|_{H^{\gamma}(\mathbb{R})} \lesssim n^{-2s+2\gamma}.$$ 

**Estimating the** $H^{\gamma-1}$ **norm of** $F_7$. Duplicating the process from the estimation of the $H^{\gamma-1}$ norm of $F_7$ we obtain

$$\|F_7\|_{H^{\gamma-1}(\mathbb{R})} \lesssim n^{-2s+2\gamma}.$$ 

Combining all of our terms for $E$ and $F$ we get the following proposition.

**Proposition 7.** Let $s > 5/2$ and $0 < \delta < 2$. Then for $\omega$ bounded and $n \gg 1$ we have

$$\|E(t)\|_{H^{\gamma}(\mathbb{R})} \lesssim n^{-r_s} \quad (6.13)$$

$$\|F(t)\|_{H^{\gamma-1}(\mathbb{R})} \lesssim n^{-\varepsilon_s} \quad (6.14)$$
with
\[ r_s = s - \delta + 1 - \gamma \quad (6.15) \]
\[ \varepsilon_s = s - \delta/2 + 1 - \gamma, \quad s > \gamma + \delta/2. \quad (6.16) \]

**Estimating the difference between approximate and actual solutions.**
Let \((u_\omega,n, \rho_\omega,n)\) be the solution to the 2-component Camassa-Holm system with initial data given by the approximate solution \((u_\omega,n, \rho_\omega,n)\) evaluated at time zero. That is \((u_\omega,n, \rho_\omega,n)\) solves the Cauchy problem
\[
\begin{align*}
\partial_t u_\omega,n + u_\omega,n \partial_x u_\omega,n + (1 - \partial_x^2)^{-1} \partial_x \left[ (u_\omega,n)^2 + \frac{1}{2} (\partial_x u_\omega,n)^2 + \frac{\sigma}{2} (\rho_\omega,n)^2 \right] &= 0 \quad (6.17) \\
\partial_t \rho_\omega,n + \partial_x (u_\omega,n \rho_\omega,n) &= 0 \quad (6.18) \\
u_\omega,n(x, 0) &= u_\omega,0^n(x) = \omega n^{-1} \tilde{\varphi}(\frac{x}{n}) + n^{-\delta/2-s} \varphi(\frac{x}{n^\delta}) \cos(n x) \quad (6.19) \\
\rho_\omega,n(x, 0) &= \rho_\omega,0^n(x) = \omega n^{-1} \tilde{\psi}(\frac{x}{n}) + n^{-\delta/2-s} \psi(\frac{x}{n^\delta}) \cos(n x). \quad (6.20)
\end{align*}
\]

Note that the initial data \((u_0^n, \rho_0^n)\) is in \(H^s \times H^{s-1}(\mathbb{R})\) since we have
\[
\begin{align*}
\|u_0^n(x)\|_{H^s(\mathbb{R})} &\simeq 1 \quad (6.21) \\
\|\rho_0^n(x)\|_{H^{s-1}(\mathbb{R})} &\simeq n^{-1}. \quad (6.22)
\end{align*}
\]

Applying Theorem 1 to the Cauchy problem \((6.17)-(6.20)\) and using estimates \((6.21)\) and \((6.22)\) for the size of the initial data, we conclude that there exists a \(T > 0\) such that for any \(\omega\) in a bounded set and \(n >> 1\) that the above IVP has a solution in
To estimate the difference between the approximate and actual solutions, we note that $v = u^{\omega,n} - u_{\omega,n}$ and $\theta = \rho^{\omega,n} - \rho_{\omega,n}$ satisfies the following initial value problem

$$
\begin{align*}
\partial_t v &= E - \frac{1}{2} \partial_x (fv) - D^{-2} \partial_x \left[ \frac{1}{2} \partial_x f \partial_x v + \frac{\sigma}{2} h \theta \right] \\
\partial_t \theta &= F - \frac{1}{2} \partial_x (vh + f\theta),
\end{align*}
$$

where $v(x,0) = \theta(x,0) = 0$, $x \in \mathbb{R}$, $t \in \mathbb{R}$, $f = u^{\omega,n} + u_{\omega,n}$, and $h = \rho^{\omega,n} + \rho_{\omega,n}$.

Furthermore, $E$ and $F$ satisfy the estimates (6.13) and (6.14) respectively.

**Lemma 31.** Let $s > 5/2$. The differences $v$ and $\theta$ satisfy

$$
\| (v(t), \theta(t)) \|_{\gamma} \lesssim n^{-r_s} + n^{-\varepsilon_s} \lesssim n^{-r_s}, \quad 0 \leq t \leq T, \quad n \gg 1,
$$

where $r_s > 0$ and $\varepsilon_s > 0$ are given by (6.15) and (6.16), and $\gamma$ is chosen so that $3/2 < \gamma + 1 < s$ and $\gamma < 1$.

**Proof.** Replicating the procedure for the estimation of the $H^\gamma$ energy of $v$ in the periodic case, we have that $v$ satisfies the differential inequality

$$
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|v(t)\|^2_{H^\gamma(\mathbb{R})} &\lesssim n^{-r_s} \|v\|_{H^\gamma(\mathbb{R})} + \|f\|_{H^s(\mathbb{R})} \|v\|^2_{H^\gamma(\mathbb{R})} \\
&\quad + \|f\|_{H^{s+1}(\mathbb{R})} \|v\|^2_{H^\gamma(\mathbb{R})} + \|h\|_{H^{\gamma}(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} \|v\|_{H^\gamma(\mathbb{R})}.
\end{align*}
$$

(6.23)

Using the Sobolev Embedding Theorem in conjunction with our solution size estimate
we have that \( \|f\|_{H^s(\mathbb{R})} \), \( \|f\|_{H^{s+1}(\mathbb{R})} \), and \( \|h\|_{H^\gamma(\mathbb{R})} \lesssim 1 \), which reduces (6.23) to
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{\gamma}(\mathbb{R})}^2 \lesssim n^{-r_s} \|v\|_{H^{\gamma}(\mathbb{R})} + \|v\|_{H^{\gamma}(\mathbb{R})}^2 + \|\theta\|_{H^{\gamma-1}(\mathbb{R})} \|v\|_{H^{\gamma}(\mathbb{R})}.
\]

Replicating the procedure for the estimation of the \( H^{\gamma-1} \) energy of \( \theta \) in the periodic case, we have that \( \theta \) satisfies the differential inequality
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}(\mathbb{R})} \lesssim n^{-\varepsilon_s} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|v\|_{H^{\gamma}(\mathbb{R})} \|h\|_{H^{\gamma}(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})}
\]
\[
+ \|f\|_{H^s(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|\partial_x f\|_{L^\infty(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})}.
\]
(6.24)

Using the Sobolev Embedding Theorem in conjunction with our solution size estimate (2.3), we have that \( \|h\|_{H^\gamma(\mathbb{R})} \), \( \|f\|_{H^s(\mathbb{R})} \) and \( \|\partial_x f\|_{L^\infty(\mathbb{R})} \) are bounded by a constant, which reduces (6.24) to
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}(\mathbb{R})} \lesssim n^{-\varepsilon_s} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|v\|_{H^{\gamma}(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|\theta\|_{H^{\gamma-1}(\mathbb{R})}^2.
\]

The differential inequalities for \( v \) and \( \theta \) now give us the following system of ODEs
\[
\frac{1}{2} \frac{d}{dt} \|v(t)\|_{H^{\gamma}(\mathbb{R})}^2 \lesssim n^{-r_s} \|v\|_{H^{\gamma}(\mathbb{R})} + \|v\|_{H^{\gamma}(\mathbb{R})}^2 + \|\theta\|_{H^{\gamma-1}(\mathbb{R})} \|v\|_{H^{\gamma}(\mathbb{R})}.
\]
(6.25)
\[
\frac{1}{2} \frac{d}{dt} \|\theta(t)\|_{H^{\gamma-1}(\mathbb{R})} \lesssim n^{-\varepsilon_s} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|v\|_{H^{\gamma}(\mathbb{R})} \|\theta\|_{H^{\gamma-1}(\mathbb{R})} + \|\theta\|_{H^{\gamma-1}(\mathbb{R})}^2.
\]
(6.26)

Carrying out differentiation on the right hand side of both (6.25) and (6.26) and
simplifying gives us

\[
\frac{d}{dt} \|v(t)\|_{H^\gamma(\mathbb{R})} \lesssim n^{-r_s} + \|(v, \theta)\|_\gamma \quad (6.27)
\]

\[
\frac{d}{dt} \|\theta(t)\|_{H^\gamma-1(\mathbb{R})} \lesssim n^{-\varepsilon_s} + \|(v, \theta)\|_\gamma. \quad (6.28)
\]

Letting \( x = \|v\|_{H^\gamma(\mathbb{R})} \) and \( y = \|\theta\|_{H^\gamma-1(\mathbb{R})} \) and adding (6.27) and (6.28) delivers us the following ODE

\[
\frac{d}{dt}(x + y) \lesssim n^{-r_s} + n^{-\varepsilon_s} + x + y.
\]

Now let \( z = x + y \) and we have

\[
\frac{dz}{dt} \lesssim n^{-r_s} + n^{-\varepsilon_s} + z.
\]

Solving the above differential inequality we have

\[
\frac{d}{dt}[e^{-t}z] \lesssim (n^{-r_s} + n^{-\varepsilon_s})e^{-t} \implies z(t) \lesssim -(n^{-r_s} + n^{-\varepsilon_s}) + z(0)e^t.
\]

Since \( z(0) = 0 \) we have that

\[
z(t) \lesssim n^{-r_s} + n^{-\varepsilon_s} \implies \|(v, \theta)\|_\gamma \lesssim n^{-r_s} + n^{-\varepsilon_s} \lesssim n^{-r_s}.
\]

This concludes the proof to Lemma 31. □

**Proof of Non-Uniform Dependence.** Consider the sequences of (actual) unique solutions to the Cauchy problem (6.17) to (6.20) given by \((u_{1,n}(x, t), \rho_{1,n}(x, t))\)
and \((u_{-1,n}(x,t), \rho_{-1,n}(x,t))\) with initial data

\[
(u_{1,n}(x,0), \rho_{1,n}(x,0)) \text{ and } (u_{-1,n}(x,0), \rho_{-1,n}(x,0))
\]

respectively. Applying Lemma \[\text{31}\] we have

\[
\|(u^{\omega,n}(t) - u_{\omega,n}(t), \rho^{\omega,n}(t) - \rho_{\omega,n}(t))\|_{\gamma} \lesssim n^{-r_s}, \quad 0 \leq t \leq T.
\]

Furthermore, from our solution size estimate, we have for \(k > s\) that

\[
\|(u_{\omega,n}(t), \rho_{\omega,n}(t))\|_k \lesssim \|(u_{\omega,n}(0), \rho_{\omega,n}(0))\|_k \\
\lesssim n^{-1} + n^{k-s} + n^{k-s-1} \\
\lesssim n^{k-s}.
\]

This implies that

\[
\|u_{\omega,n}(t)\|_{H^k(\mathbb{R})} \lesssim \|(u_{\omega,n}(0), \rho_{\omega,n}(0))\|_k \lesssim n^{k-s}, \\
\|\rho_{\omega,n}(t)\|_{H^{k-1}(\mathbb{R})} \lesssim \|(u_{\omega,n}(0), \rho_{\omega,n}(0))\|_k \lesssim n^{k-s}.
\]

Thus, by utilizing the triangle inequality, we obtain

\[
\|u^{\omega,n}(t) - u_{\omega,n}(t)\|_{H^k(\mathbb{R})} \lesssim n^{-1+\delta/2} + n^{k-s} \lesssim n^{k-s}, \\
\|\rho^{\omega,n}(t) - \rho_{\omega,n}(t)\|_{H^{k-1}(\mathbb{R})} \lesssim n^{-1+\delta/2} + n^{k-s} + n^{k-s-1} \lesssim n^{k-s}.
\]
Now we wish to use the following interpolation lemma.

**Lemma 32.** Let \( f \in H^s \) and \( s_1 < s < s_2 \). Then

\[
\|f\|_{H^s} \leq \|f\|_{H^{s_1}}^{\frac{s_2-s}{s_2-1}} \|f\|_{H^{s_2}}^{\frac{s-s_1}{s_2-1}}.
\]

Let \( v(t) = u_{\omega,n}^{\omega}(t) - u_{\omega,n}(t) \) and \( \theta(t) = \rho_{\omega,n}^{\omega}(t) - \rho_{\omega,n}(t) \). Interpolating between the \( H^\gamma(\mathbb{R}) \) norm and the \( H^k(\mathbb{R}) \) norm to get an \( H^s(\mathbb{R}) \) estimate for \( v(t) \) and between the \( H^{\gamma-1}(\mathbb{R}) \) norm and the \( H^{k-1}(\mathbb{R}) \) norm to get an \( H^{s-1}(\mathbb{R}) \) estimate for \( \theta(t) \) where \( k = [s] + 2 \) yields the following estimate:

\[
\|v(t)\|_{H^s(\mathbb{R})} \leq \|v(t)\|_{H^\gamma(\mathbb{R})}^{\frac{k-s}{k-\gamma}} \|v(t)\|_{H^k(\mathbb{R})} \lesssim (n^{-r_s})^{\frac{k-s}{k-\gamma}} (n^{k-s})^{\frac{s-\gamma}{k-\gamma}} \lesssim n^{-\beta_s}
\]

\[
\|\theta(t)\|_{H^{s-1}(\mathbb{R})} \leq \|\theta(t)\|_{H^{\gamma-1}(\mathbb{R})}^{\frac{k-s}{k-\gamma}} \|\theta(t)\|_{H^{k-1}(\mathbb{R})} \lesssim (n^{-r_s})^{\frac{k-s}{k-\gamma}} (n^{k-s})^{\frac{s-\gamma}{k-\gamma}} \lesssim n^{-\beta_s}
\]

where we have that

\[
\beta_s = \frac{(1 - \delta)(k - s)}{k - \gamma}.
\]

Here we note that \( \beta_s > 0 \) for \( \delta \in (0, 1) \) and \( k = [s] + 2 \).

Now at time \( t = 0 \), we have

\[
\|u_{1,n}(0) - u_{-1,n}(0)\|_{H^s(\mathbb{R})} = \|2n^{-1}\hat{\phi}(\frac{x}{n^\delta})\|_{H^s(\mathbb{R})} \simeq n^{-1+\delta/2} \to 0 \text{ as } n \to \infty
\]

\[
\|\rho_{1,n}(0) - \rho_{-1,n}(0)\|_{H^{s-1}(\mathbb{R})} = \|2n^{-1}\hat{\psi}(\frac{x}{n^\delta})\|_{H^{s-1}(\mathbb{R})} \simeq n^{-1+\delta/2} \to 0 \text{ as } n \to \infty.
\]
However, at time $t > 0$ we have
\[
\|u_{1,n}(t) - u_{-1,n}(t)\|_{H^s(\mathbb{R})} \geq \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(\mathbb{R})} - \|u^{1,n}(t) - u_{1,n}(t)\|_{H^s(\mathbb{R})}
- \|u^{-1,n}(t) - u_{-1,n}(t)\|_{H^s(\mathbb{R})}.
\] (6.29)

and
\[
\|\rho_{1,n}(t) - \rho_{-1,n}(t)\|_{H^{s-1}(\mathbb{R})} \geq \|\rho^{1,n}(t) - \rho^{-1,n}(t)\|_{H^{s-1}(\mathbb{R})} - \|\rho^{1,n}(t) - \rho_{1,n}(t)\|_{H^{s-1}(\mathbb{R})}
- \|\rho^{-1,n}(t) - \rho_{-1,n}(t)\|_{H^{s-1}(\mathbb{R})}.
\] (6.30)

Adding (6.29) and (6.30) we have that
\[
\|(u_{1,n}(t) - u_{-1,n}(t), \rho_{1,n}(t) - \rho_{-1,n}(t))\|_{s} \geq \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(\mathbb{R})} - cn^{-\beta_s}
+ \|\rho^{1,n}(t) - \rho^{-1,n}(t)\|_{H^{s-1}(\mathbb{R})} - an^{-\beta_s}.
\]

Taking the limit infimum of both sides gives us
\[
\lim_{n \to \infty} \inf (\|(u_{1,n}(t) - u_{-1,n}(t), \rho_{1,n}(t) - \rho_{-1,n}(t))\|_{s})
\geq \lim_{n \to \infty} \inf (\|(u^{1,n}(t) - u^{-1,n}(t), \rho^{1,n}(t) - \rho^{-1,n}(t))\|_{s}).
\]

Hence, to finish the argument for the non-periodic case of Theorem 2 we need only find a lower bound for the difference of known approximate solutions $u^{1,n}(t) - u^{-1,n}(t)$ and $\rho^{1,n}(t) - \rho^{-1,n}(t)$. 

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Using the identity \( \cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \) yields

\[
\begin{align*}
  u^{1,n}(t) - u^{-1,n}(t) &= u_{l,1,n}(t) - u_{l,-1,n}(t) + 2n^{-\delta/2-s} \varphi\left(\frac{x}{n^\delta}\right) \sin(nx) \sin(t), \\
  \rho^{1,n}(t) - \rho^{-1,n}(t) &= \rho_{l,1,n}(t) - \rho_{l,-1,n}(t) + 2n^{-\delta/2-s} \psi\left(\frac{x}{n^\delta}\right) \sin(nx) \sin(t).
\end{align*}
\]

Therefore,

\[
\begin{align*}
  \|u^{1,n}(t) - u^{-1,n}(t)\|_{H^s(\mathbb{R})} &\geq 2n^{-\delta/2-s} \|\varphi\left(\frac{x}{n^\delta}\right) \sin(nx)\|_{H^s(\mathbb{R})} |\sin(t)| \\
  &- \|u_{l,1,n}(t)\|_{H^s(\mathbb{R})} - \|u_{l,-1,n}(t)\|_{H^s(\mathbb{R})} \\
  &\gtrsim 2n^{-\delta/2-s} \|\varphi\left(\frac{x}{n^\delta}\right) \sin(nx)\|_{H^s(\mathbb{R})} |\sin(t)| - n^{-1+\delta/2} \quad (6.31)
\end{align*}
\]

and

\[
\begin{align*}
  \|\rho^{1,n}(t) - \rho^{-1,n}(t)\|_{H^{s-1}(\mathbb{R})} &\geq 2n^{-\delta/2-s} \|\psi\left(\frac{x}{n^\delta}\right) \sin(nx)\|_{H^{s-1}(\mathbb{R})} |\sin(t)| \\
  &- \|\rho_{l,1,n}(t)\|_{H^{s-1}(\mathbb{R})} - \|\rho_{l,-1,n}(t)\|_{H^{s-1}(\mathbb{R})} \\
  &\gtrsim 2n^{-\delta/2-s} \|\psi\left(\frac{x}{n^\delta}\right) \sin(nx)\|_{H^{s-1}(\mathbb{R})} |\sin(t)| - n^{-1+\delta/2}. \quad (6.32)
\end{align*}
\]

Taking the limit infimum of \((6.31) + (6.32)\) and using Lemma 28 yields

\[
\begin{align*}
  \lim_{n \to \infty} \inf \left( \|u^{1,n}(t) - u^{-1,n}(t), \rho^{1,n}(t) - \rho^{-1,n}(t)\|_s \right) \gtrsim |\sin(t)|, \quad (6.33)
\end{align*}
\]

which concludes our proof of the non-periodic case of Theorem 2. \(\square\)


CHAPTER 7

HÖLDER CONTINUITY

Let $s > 5/2$. From Theorem 1, we have that for any initial data $(u_0, \rho_0) \in B(0, \lambda)$, there exists a unique solution $(u, \rho) \in C([0, T]; H^s \times H^{s-1})$ to (2.1) for some $T_0 > 0$ such that the lifespan

$$T_0 \geq \frac{1}{2c_s \|(u_0, \rho_0)\|_s} \geq \frac{1}{2c_s \lambda} \equiv T. \quad (7.1)$$

This allows us to pick a lifespan which will hold for any choice of initial data in $B(0, \lambda)$. Let $(w, \phi)$ be another solution to (1.3) with initial data $(w_0, \phi_0) \in B(0, \lambda)$. As we showed in uniqueness, we have that the $H^r$-energy of the difference $v = u - w$ is given by (3.79) and the $H^{r-1}$-energy of the difference $\theta = \rho - \phi$ is (3.82)+(3.83).

7.1. Lipschitz Continuity in $\mathcal{I}_1$. Let $1/2 < r \leq s - 1$ and $s + r \geq 2$. The estimate of (3.79) is the same as in the proof of uniqueness where $f = u + w$. The estimates (3.82) and (3.83) also follow from the proof of uniqueness. The resulting energy estimates yield

$$\|(v(t), \theta(t))\|_r \lesssim e^{csT} \|(v(0), \theta(0))\|_r.$$
Hence, we have Lipschitz continuity in $\mathcal{I}_1$.

7.2. Hölder Continuity in $\mathcal{I}_2$. Let $s - 1 < r < s$ and $s > 5/2$. By interpolation, we have

$$
\|v(t)\|_{H^r} \lesssim \|v(t)\|_{H^{s-1}}^{\frac{s-r}{s-1}} \|v(t)\|_{H^{r}}^{\frac{r-(s-1)}{s-1}} \lesssim \|v(t)\|_{H^{s-1}}^{s-r},
$$

(7.2)

$$
\|\theta(t)\|_{H^{r-1}} \lesssim \|\theta(t)\|_{H^{s-2}}^{\frac{s-1-(r-1)}{s-2}} \|\theta(t)\|_{H^{r-1}}^{\frac{r-1-(s-2)}{s-2}} \lesssim \|\theta(t)\|_{H^{s-2}}^{s-r}.
$$

(7.3)

since we have (2.3) and $(v(0), \theta(0)) \in B(0, 2\lambda)$. As $(s - 1, s) \in \mathcal{I}_1$, our Lipschitz continuity from $\mathcal{I}_1$ implies

$$
\|v(t)\|_{H^r} \lesssim \|v(t)\|_{H^{s-1}}^{s-r} \lesssim \|v(t)\|_{H^{r}}^{s-r} \lesssim \|(v(0), \theta(0))\|_{H^r}^{s-r},
$$

(7.4)

$$
\|\theta(t)\|_{H^{r-1}} \lesssim \|\theta(t)\|_{H^{s-2}}^{s-r} \lesssim \|\theta(t)\|_{H^{r-1}}^{s-r} \lesssim \|(v(0), \theta(0))\|_{H^r}^{s-r}.
$$

(7.5)

Adding (7.4) and (7.5) together gives us

$$
\|(v(t), \theta(t))\|_{H^r}^{s-r} \lesssim \|(v(0), \theta(0))\|_{H^r}^{s-r}.
$$

Thus, we have that the data-to-solution map is Hölder continuous in the $H^r \times H^{r-1}$ norm with exponent $s - r$ when $(s, r) \in \mathcal{I}_2$.

This concludes our proof of Theorem 3. □
CHAPTER 8

NUMERICS

Introduction to the Spectral Method

Spectral methods are a class of numerical methods used in applied mathematics to numerically solve differential equations. The idea behind all spectral methods is to write the solution of a particular PDE as a sum of basis functions. For example, we will explore spectral methods which use sinusoids as basis functions, and therefore, the solution will be represented by a Fourier series.

The defining characteristic of spectral methods, when compared to other methods (in particular finite element methods), is that spectral methods are “global”. We take as basis functions, functions which are nonzero over the entire domain. Thus, the solution is estimated over the entire domain using the same set of basis functions. Other methods are “local”, in that they split the domain into sub-domains, and treat each sub-domains separately (though not completely independently).

As a result of the “global” approach, spectral methods converge faster than any other method when the solution is smooth. Their rate of convergence is sometimes referred to as infinite or exponential. Moreover, we can use these methods to solve a broad range of differential equations, including ODEs and PDEs. When we consider
time dependent PDEs, we will sometimes treat the time variable differently than the
spacial variable, only using the spectral method for one of the two variables.

These methods date back to 1969, and have been developed for periodic geometric
boundary conditions using the Fourier series, polynomial spectral methods for
more general boundary conditions or pseudospectral methods for nonlinear problems.
These methods are better than finite element or finite difference methods for simple
domains and smooth solutions. However, they are not well suited for complex domains
or discontinuities. We will explore these phenomenon in the following subsections.

All of the information contained in this document was taken from Trefethen [55].

**Spectral Method**

We begin with an overview of the Fourier spectral method. Suppose \( u(x) \) is a smooth
2\( \pi \)–periodic function in \( x \). Using the *Inverse Fourier Transform* we have

\[
{u(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{ikx} \hat{u}(k)}
\]  

(8.1)

Then, we have that \( u(x) \) is written as an infinite sum of basis functions.

**STEP 1.** We will throw away the high frequencies (*discrete inverse Fourier trans-
form*). We estimate \( u(x) \) via a finite number of basis functions. Thus we will take
\( N > 1 \) an *even* integer, and take as our *approximation*

\[
{u(x) \approx \frac{1}{2\pi} \sum_{k=-N/2+1}^{N/2} e^{ikx} \hat{u}(k)}
\]  

(8.2)

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Vector Notation. Another way to express the above is in the following vector notation. We will always use capital letters for vectors and subscripts will denote components. We define

\[
K \doteq \begin{pmatrix}
-\frac{N}{2} + 1 \\
-\frac{N}{2} + 2 \\
\vdots \\
\frac{N}{2}
\end{pmatrix}, \quad \hat{U} \doteq \begin{pmatrix}
\hat{u}(-\frac{N}{2} + 1) \\
\hat{u}(-\frac{N}{2} + 2) \\
\vdots \\
\hat{u}\left(\frac{N}{2}\right)
\end{pmatrix},
\] (8.3)

Thus for example, we see that \(K_1 = -\frac{N}{2}\), and \(K_N = \frac{N}{2} - 1\). With this notation, we can express \(u^N\) as

\[
u(x) \approx \frac{1}{2\pi} \sum_{n=1}^{N} e^{iK_n x} \hat{u}(K_n) = \frac{1}{2\pi} \sum_{n=1}^{N} e^{iK_n x} \hat{U}_n
\] (8.4)

STEP 2. Analytically, we know that the Fourier transform is expressed as the integral

\[
\hat{u}(k) = \int_{0}^{2\pi} e^{-ikx} u(x) dx
\] (8.5)

the question is how this is done via the computer.

Our approximation for this integral will be a Riemann Sum! if we break the domain into \(N\) intervals we have
\[
\int_0^{2\pi} e^{-ikx} u_0(x) dx \approx \frac{2\pi}{N} \sum_{n=0}^{N-1} u\left(\frac{2\pi n}{N}\right) \cdot e^{-ik\left[\frac{2\pi n}{N}\right]} \quad (8.6)
\]

which is easily computable!

**Vector Notation.** We will take our set of points in the domain to be

\[
X = \begin{pmatrix}
0 \\
1 \cdot \frac{2\pi}{N} \\
2 \cdot \frac{2\pi}{N} \\
\vdots \\
(N-1) \cdot \frac{2\pi}{N}
\end{pmatrix}, \quad U = \begin{pmatrix}
u(X_1) \\
u(X_2) \\
\vdots \\
u(X_N)
\end{pmatrix}, \quad (8.7)
\]

Then our Riemann sum can be rewritten as

\[
\int_0^{2\pi} e^{-ikx} u_0(x) dx \approx \frac{2\pi}{N} \sum_{n=1}^{N} U_n \cdot e^{-i\hat{K} \cdot X_n}. \quad (8.8)
\]

Therefore, we have for each element \(K_m\)

\[
\hat{U}_m = \frac{2\pi}{N} \sum_{n=1}^{N} U_n \cdot e^{-iK_m \cdot X_n}. \quad (8.9)
\]

**STEP 3.** Thus we have the following approximation
\[ u(x) \approx \frac{1}{N} \sum_{m=1}^{N} e^{iK_m x} \sum_{n=1}^{N} e^{-iK_m \cdot X_n} U_n. \]  

(8.10)

Summary.

1. Given our domain \([0, 2\pi]\), we break it into \(N\) evenly spaces points, and we label these points the vector \(X\).

2. We then evaluate our function, \(u\), at each gridpoint in \(X\), and label the approximate vector \(U\).

3. We truncate the integers, and label the truncated integers \(K\), so that given \(U\), we can estimate the Fourier transform of \(U\), labeled \(\hat{U}\), via step 2.

4. Given \(\hat{U}\), we can estimate \(u\) via the inverse Fourier transform in step 1.

From what we have derived, we see that we can go back and forth between the phase space and the base space via the discrete Fourier transform and it’s discrete inverse. We additionally note that these computations have been optimized for efficiency, and the optimized versions are known as the fast Fourier transform, and inverse fast Fourier transform respectively. They are built into Matlab, or can be downloaded for other programming languages.
Spectral Method for CH2 system

The i.v.p. for the 2-component Camassa-Holm (CH2) system is

\[
\begin{cases}
  u_t + uu_x + \partial_x(1 - \partial_x^2)^{-1} \left[ u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\rho^2 \right] = 0, \\
  \rho_t + (u\rho)_x = 0 \\
  u(0, t) = u(2\pi, t) \\
  \rho(0, t) = \rho(2\pi, t) \\
  u(x, 0) = f(x) \\
  \rho(x, 0) = g(x).
\end{cases}
\]  

(8.11)

Taking the Fourier transform we obtain

\[
\begin{align*}
\hat{u}_t + \frac{1}{2}ik\hat{u}^2 + \frac{ik}{1 + k^2} \left[ \hat{u}^2 + \frac{1}{2}\hat{u}_x^2 + \frac{1}{2}\hat{\rho}^2 \right] &= 0 \quad (8.12) \\
\hat{u}_t + \frac{1}{2}ik\mathcal{F}((\mathcal{F}^{-1}(\hat{u}))^2) \\
+ \frac{ik}{1 + k^2} \left[ \mathcal{F}((\mathcal{F}^{-1}(\hat{u}))^2) + \frac{1}{2}\mathcal{F}((\mathcal{F}^{-1}(ik\hat{u}))^2) + \frac{1}{2}\mathcal{F}((\mathcal{F}^{-1}(\hat{\rho}))^2) \right] &= 0 \quad (8.13) \\
\hat{\rho}_t + ik\hat{u}\hat{\rho} &= 0 \quad (8.14) \\
\hat{\rho}_t + ik\mathcal{F}((\mathcal{F}^{-1}(\hat{u})\mathcal{F}^{-1}(\hat{\rho})) &= 0 \quad (8.15)
\end{align*}
\]

For simplicity, we define \( \hat{v} = \mathcal{F}((\mathcal{F}^{-1}(\hat{u}))^2) \), \( \hat{w} = \frac{1}{2}\mathcal{F}((\mathcal{F}^{-1}(ik\hat{u}))^2) \), \( \hat{y} = \frac{1}{2}\mathcal{F}((\mathcal{F}^{-1}(\hat{\rho}))^2) \)
and \( \hat{z} = \mathcal{F}((\mathcal{F}^{-1}(\hat{u})\mathcal{F}^{-1}(\hat{\rho})) ). \) Then our system becomes

\[
\hat{u}_t = -\frac{1}{2}ik\hat{v} - \frac{ik}{1+k^2}\hat{\omega} - \frac{ik}{1+k^2}\hat{\nu} - \frac{ik}{1+k^2}\hat{y} \tag{8.16}
\]

\[
\hat{\rho}_t = -ik\hat{z}. \tag{8.17}
\]

We discretize by restricting \( k \) to the set of integers \( \{-N/2, \ldots, N/2 - 1\} \) and \( x \) to the set of points \( \{0, 2\pi/N, \ldots, (N-1)2\pi/N\} \). Thus, we have a set of \( N \) ODEs for each component which we can move forward using Euler’s method:

1. Begining at time \( t = 0 \), we solve the system of ODEs at time \( \Delta t \) according to the rule

\[
\hat{u}^{(k)}(\Delta t) = \hat{u}^{(k)}(0) - \Delta t \frac{1}{2}ik\hat{v}^{(k)}(0) - \Delta t \frac{ik}{1+k^2}\hat{\omega}^{(k)}(0) - \Delta t \frac{ik}{1+k^2}\hat{\nu}^{(k)}(0) - \Delta t \frac{ik}{1+k^2}\hat{y}^{(k)}(0)
\]

\[
\hat{\rho}^{(k)}(\Delta t) = \hat{\rho}^{(k)}(0) - \Delta t ik\hat{z}^{(k)}(0).
\]

2. Now we calculate \( \hat{v}(\Delta t), \hat{w}(\Delta t), \hat{y}(\Delta t) \) and \( \hat{z}(\Delta t) \).

3. Now we step forward to time \( 2\Delta t \), using the rule

\[
\hat{u}^{(k)}(t + \Delta t) = \hat{u}^{(k)}(t) - \Delta t \frac{1}{2}ik\hat{v}^{(k)}(t) - \Delta t \frac{ik}{1+k^2}\hat{\omega}^{(k)}(t) - \Delta t \frac{ik}{1+k^2}\hat{\nu}^{(k)}(t) - \Delta t \frac{ik}{1+k^2}\hat{y}^{(k)}(t)
\]

\[
\hat{\rho}^{(k)}(t + \Delta t) = \hat{\rho}^{(k)}(t) - \Delta t ik\hat{z}^{(k)}(t).
\]

Below is our MATLAB code and \( u(x, t) \) and \( \rho(x, t) \) at time \( t = 1 \) with initial data \( u(x, 0) = \sin x \) and \( \rho(x, 0) = \cos x \). See Figure 8.1 for the results of our code.
N = 2^7; dt = 2^{(-10)}/3;
x = (2*pi/N)*(0:N-1)'; tmax =5;

u = 1*sin(x) rho = 1*cos(x);

uhat = fft(u); rhohat = fft(rho);
k = [0:N/2 -N/2+1:-1]'; ik2 = -1*k.^2;

for n = 1:nmax
    vhat = fft(real(ifft(uhat)).^2);
    what = 0.5*fft(real(ifft(1i*k.*uhat)).^2);
    yhat = 0.5*fft(real(ifft(rhohat)).^2);
    zhat = fft(real(ifft(uhat)).*real(ifft(rhohat)));
    uhat = uhat -0.5i*dt*k.*vhat - dt*(1i*k./(1+k.^2)).*(vhat + what + yhat);
    rhohat = rhohat -1i*dt*k.*zhat;
end

**Further exploration:** We could develop a more sophisticated numerical scheme in order to increase step size and time interval to exhibit blow up of the solution. Since we know that for this particular initial data \((1 - \partial_x^2)u_0\) changes sign, we will have such blow-up. One possibility is to use finite element methods and another is to keep the spectral method but improve upon it by utilizing something other than Euler’s method. Furthermore, we could look at creating a numerics scenario that will visualize the non-uniform dependence and Hölder continuity upon the initial data. Examining this behavior on the torus, we can keep the spectral method, but on the real line,
we need a pseudo-spectral method or a spectral collocation method to analyze the aforementioned plus global solutions and blow-up scenarios on the real line.
Figure 8.1. Graphs generated by the Spectral Method.
CHAPTER 9

CONCLUDING REMARKS FOR FUTURE RESEARCH

In this section, we make a few comments regarding future research on the CH2 system. The first problem that we may examine is extending well-posedness result to classical solutions. Essentially, we would move to the rougher space of functions $C^1$. In order to make this extension, we will look at the paper on classical solutions for the Camassa-Holm equation by Misiolek in [46]. Throughout his paper he utilizes the ideas from Ebin and Marsden [13] and Arnold [1]. These aforementioned papers should also be read so as to draw proper mathematical machinery and inspiration.

It would also benefit us to see if the Sobolev index $s > 5/2$ may be lowered to $s > 3/2$. In fact, what may be interesting is to see what happens at $s = 3/2$ or whatever the critical Sobolev index is (which should be found by using a scaling argument on the CH2 system). For $s > 3/2$, we still need to investigate Gui and Liu’s paper [20] and make sure the result is sound. If so, we may proceed to investigating a well-posedness result at the critical index $s = 3/2$ but in Besov spaces. Danchin’s paper [10] show’s well-posedness for the Camassa-Holm (CH) equation in Besov spaces for $s = 3/2$ and may provide us with some insight as to how it may be done for CH2.
Last, but not least, we should examine the geometric extensions of the CH2 system. Misiolek provides us with such an extension for the CH equation in [45]. A review of this paper may aid us in understand the paper of Escher, Kohlmann and Lenells [15] where they find a geometric extension for the CH2 system. The purpose would not be for generating any new results, but to simply expand our mathematical knowledge from the analytical to the geometrical.
Definition 2. Let $E, F$ be Banach spaces, $\Omega \subset E$ be an open subset and $f : \Omega \to F$. We say that $f$ is differentiable at $x_0 \in \Omega$ if there exists a linear map $u : E \to F$ such that
\[
\lim_{x \to x_0} \frac{\|f(x) - f(x_0) - u(x - x_0)\|_F}{\|x - x_0\|_E} = 0
\]
u is called the derivative of $f$ at the point $x_0$ and is usually written as $f'(x_0)$ or $Df(x_0)$.

Consider a function $f : (a, b) \to \mathbb{R}$. One can think of the derivative of $f$ at a point $x_0 \in (a, b)$ as the slope $m \in \mathbb{R}$ of the line $x \mapsto f(x_0) + m(x - x_0)$ that lies tangent to the graph of $f$ at $x_0$. Here, we generalize this notion in order to define the derivative of a function in Banach spaces. In other words, if $f$ is differentiable at $x_0$, then there exists a linear map $Df(x_0)$ which is tangent to $f$ at $x_0$.

Proposition 8. If $f : \Omega \to F$ is continuous and differentiable at $x_0$, then we have that $u = Df(x_0)$ is unique and continuous.
Proof: We shall first prove uniqueness. For ease of notation, we will denote a linear function \( \Lambda : X \to Y \) tangent to \( f \) at \( x_0 \) as the derivative. Let \( \Lambda_1 \) be another such function. Then we have

\[
\lim_{y \to 0} \frac{\|(\Lambda - \Lambda_1)(y)\|}{\|y\|} = \lim_{x \to x_0} \frac{\|\Lambda(x - x_0) - \Lambda_1(x - x_0)\|}{\|x - x_0\|}
\]

We add zero and then use the triangle inequality to find

\[
\lim_{y \to 0} \frac{\|(\Lambda - \Lambda_1)(y)\|}{\|y\|} = \lim_{x \to x_0} \frac{\|\Lambda(x - x_0) - \Lambda_1(x - x_0) \pm f(x) \pm f(x_0)\|}{\|x - x_0\|}
\]

\[
\leq \lim_{x \to x_0} \frac{\|f(x) - f(x_0) - \Lambda(x - x_0)\| + \|f(x) - f(x_0) - \Lambda_1(x - x_0)\|}{\|x - x_0\|}
\]

\[
= 0.
\]

By definition of the operator norm, we have

\[
\lim_{y \to 0} \frac{\|(\Lambda - \Lambda_1)(y)\|}{\|y\|} = \|\Lambda - \Lambda_1\|.
\]

Thus, we have that \( \Lambda = \Lambda_1 \), which proves uniqueness.

Now we prove continuous. Since \( f' \) is a linear function, it suffices to show that \( f' \) is continuous at 0. Let \( \varepsilon > 0 \). Since \( f \) is continuous at \( x_0 \) and \( f' \) is tangent to \( f \) at \( x_0 \), there is an \( r \in (0, 1) \) such that

\[
\|x_0 + t - x_0\| = \|t\| \leq r \Rightarrow \|f(x_0 + t) - f(x_0)\| \leq \frac{\varepsilon}{2}
\]
and implies

$$\|f(x_0 + t) - f(x_0) - f'(t)\| \leq \frac{\varepsilon}{2} \|t\| \leq \frac{\varepsilon}{2}.$$  

This implies that for $\|t\| \leq r$ we have

$$\|\Lambda(t)\| = \|f(x_0 + t) - f(x_0) - \Lambda(t) - (f(x_0 + t) - f(x_0))\|.$$  

We then use the triangle inequality to get

$$\|\Lambda(t)\| \leq \|f(x_0 + t) - f(x_0) - \Lambda(t)\| + \|f(x_0 + t) - f(x_0)\|$$

$$< \varepsilon.$$  

Thus, we have found that $\Lambda$ is a linear map which is continuous at 0. Therefore, $u \in \mathcal{L}(X,Y)$. □

**Proposition 9** (Chain Rule). Suppose $U \subset X$ and $V \subset Y$. Let $f \in C(U; Y)$ and $g \in C(V; Y)$ such that $f(U) \subset V$. Also, suppose $f$ is differentiable at $x_0 \in U$ and $g$ is differentiable at $y_0 = f(x_0) \in V$. Then the mapping $g \circ f$ is differentiable at $x_0$. The derivative is given by

$$(g \circ f)'(x_0) = g'(y_0) \circ f'(x_0).$$

**Proof:** By assumption, given $\varepsilon \in (0, 1)$, there is an $r > 0$ such that for $\|s\| \leq r$
and $\|t\| \leq r$, we have

\[ f(x_0 + s) = f(x_0) + f'(x_0) \cdot s + o_1(s) \]
\[ g(y_0 + t) = g(y_0) + g'(y_0) \cdot t + o_2(t) \]

where $\|o_1(s)\| \leq \varepsilon \|s\|$ and $\|o_2(t)\| \leq \varepsilon \|t\|$. We write

\[ g(f(x_0 + s)) = g(f(x_0) + f'(x_0) \cdot s + o_1(s)) \]
\[ = g(y_0) + g'(y_0) \cdot (f'(x_0) \cdot s + o_1(s)) + o_2(f'(x_0) \cdot s + o_1(s)) \]
\[ = g(y_0) + g'(y_0) \cdot (f'(x_0) \cdot s + g'(y_0) \cdot o_1(s) + o_2(f'(x_0) \cdot s + o_1(s))). \]

(A.1)

We know that $f'(x_0)$ and $g'(y_0)$ are continuous linear mappings. Therefore, there exist constants $a, b$ such that

\[ \|f'(x_0) \cdot s\| \leq a \|s\| \text{ and } \|g'(y_0) \cdot t\| \leq b \|t\|. \]

Hence, for all $\|s\| \leq r$, we have

\[ \|f'(x_0) \cdot s + o_1(s)\| \leq \|f'(x_0) \cdot s\| + \|o_1(s)\| \leq (a + \varepsilon) \|s\| \leq (a + 1) \|s\|. \]

As a result, there exists $r' > 0$ such that $\|s\| \leq \min\left\{ r', \frac{r}{a + 1} \right\}$, we have

\[ \|o_2(f'(x_0) \cdot s + o_1(s))\| \leq \varepsilon \|f'(x_0) \cdot s + o_1(s)\| \leq \varepsilon (a + 1) \|s\|. \]

(A.2)
Moreover,

\[ \|g'(y_0) \cdot o_1(s)\| \leq b\|o_1(s)\| \leq b\varepsilon\|s\|. \]  

(A.3)

We now return our attention to (A.1). Using (A.2) and (A.3), we now have

\[ g(f(x_0 + s)) \leq g(y_0) + g'(y_0) \cdot (f'(x_0) \cdot s) + o_3(s), \]

where

\[ \|o_3(s)\| \leq b\varepsilon\|s\| + (a + 1)\varepsilon\|s\| = (a + b + 1)\varepsilon\|s\|. \]

This yields

\[ (g \circ f)'(x_0) = g'(y_0) \circ f'(x_0), \]

which completes our proof of the Chain Rule.

One of the applications of the Chain Rule is the linearity of differentiation.

**Lemma 33** (Linearity of Differentiation). For \( a, b \in \mathbb{R} \) and \( f, g \in C(U;Y) \), the differential \( D \) is linear; i.e.

\[ D(af + bg) = aDf + bDg. \]

**Theorem 8** (Mean Value Theorem). Let \( I = [\alpha, \beta] \subset \mathbb{R} \), \( F \) a Banach space and \( f : I \to F \), \( \phi : I \to \mathbb{R} \) be continuous maps. Suppose there is a denumerable subset \( D = \{d_n\} \subset I \) such that for each \( x \in I - D \) both \( f \) and \( \phi \) have a derivative at \( x \) and
that $\|f\|_F \leq \phi'(x)$. Then $\|f(\beta) - f(\alpha)\|_F \leq \phi(\beta) - \phi(\alpha)$.

Proof. It suffices to show that

$$\forall \epsilon > 0, \|f(\beta) - f(\alpha)\|_F \leq \phi(\beta) - \phi(\alpha) + \epsilon(\beta - \alpha + 1).$$

So let's define

$$S = \left\{ y \in I \mid \forall x \in [\alpha, y), \|f(\beta) - f(\alpha)\|_F \leq \phi(\beta) - \phi(\alpha) + \epsilon(x - \alpha) + \epsilon \sum_{d_n < x} 2^{-n} \right\}.$$

We know that $S$ is nonempty since $\alpha \in S$. Furthermore, if $x < y$ and $y \in S$ then $x \in S$ by construction.

Now set $\eta = \sup S$. Then by our two previous comments we have that $S = [\alpha, \eta]$. Also, by the continuity of $f$ and $\phi$, we have

$$\|f(\eta) - f(\alpha)\|_F \leq \phi(\eta) - \phi(\alpha) + \epsilon(\eta - \alpha) + \epsilon \sum_{d_n < \eta} 2^{-n}.$$ 

Now we want to show that $\eta = \beta$. So suppose to the contrary that $\eta < \beta$. Then we must consider two cases.

**Case 1: $\eta \notin D$.** Then by the continuity of $f$ and $\phi$, we can find a subset of $I$, $[\eta, \eta + \delta)$ for some $\delta > 0$, such that for all $y \in [\eta, \eta + \delta)$ we have

1. $\|f(y) - f(\eta) - f'(\eta)(y - \eta)\|_F \leq (\epsilon/2)(y - \eta)$.
2. $|\phi(y) - \phi(\eta) - \phi'(\eta)(y - \eta)| \leq (\epsilon/2)(y - \eta)$. 

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Now by using the triangle inequality on (2) we have

\[ \phi'(\eta)(y - \eta) \leq (\epsilon/2)(y - \eta) + \phi(y) - \phi(\eta). \]

Similarly, using the triangle inequality on (1) gives us

\[ \|f(y) - f(\eta)\| \leq \|f'(\eta)\|(y - \eta) + (\epsilon/2)(y - \eta) \]
\[ \leq \phi'(\eta)(y - \eta) + (\epsilon/2)(y - \eta) \]
\[ \leq \phi(y) - \phi(\eta) + \epsilon(y - \eta). \]

So it follows that

\[ \|f(y) - f(\alpha)\| \leq \|f(y) - f(\eta)\| + \|f(\eta) - f(\alpha)\| \]
\[ \leq \phi(y) - \phi(\eta) + \epsilon(y - \eta) + \phi(\eta) - \phi(\alpha) + \epsilon(\eta - \alpha) + \sum_{d_n < \eta} 2^{-n} \]
\[ \leq \phi(y) - \phi(\alpha) + \epsilon(y - \alpha) + \sum_{d_n < y} 2^{-n}. \]

Hence, \( y \in S \), which gives us our contradiction since \( y > \eta \).

**Case 2:** \( \eta \in D \). So for some \( k \) we have that \( \eta = d_k \). Now from the continuity of \( f \) and \( \phi \), we have that for some \( \delta > 0 \) that for all \( y \in [\eta, \eta + \delta) \subset I \)

\[ \|f(y) - f(\eta)\| \leq \frac{\epsilon}{2} \cdot \frac{1}{2^k} \text{ and } |\phi(y) - \phi(\eta)| \leq \frac{\epsilon}{2} \cdot \frac{1}{2^k}. \]
It follows that

\[ \|f(y) - f(\eta)\|_F \leq \|f(y) - f(\alpha)\|_F + \|f(\eta) - f(\alpha)\|_F \]
\[ \leq \frac{\epsilon}{2} \cdot \frac{1}{2^k} + \phi(\eta) - \phi(\alpha) + \epsilon(\eta - \alpha) + \epsilon \sum_{d_n < \eta} 2^{-n} \]
\[ \leq \phi(y) - \phi(\alpha) + \epsilon(\eta - \alpha) + \epsilon \sum_{d_n < \eta} 2^{-n} \]
\[ \leq \phi(y) - \phi(\alpha) + \epsilon(y - \alpha) + \epsilon \sum_{d_n < y} 2^{-n}. \]

This, however, implies that \(y \in S\), which gives us our contradiction. This completes our proof. \(\square\)

An important case of the Mean Value Theorem is when \(\phi(x) = M(x - a)\), where \(M > 0\). This is stated as the following lemma.

**Lemma 34.** If there exists a countable set \(D\) of \(I\) such that for each \(x \in I - D\), \(f\) has at \(x\) a derivative with respect to \(I\) such that \(\|f'(x)\| \leq M\), then

\[ \|f(b) - f(a)\| \leq M(b - a). \]

An application of the Mean Value Theorem is the following corollary.

**Corollary 1.** Let \(X, Y\) be Banach spaces, \(U\) an open subset of \(X\), \(f : U \to Y\) continuously, and \(f' = 0\) at every point of \(U\), then \(f\) is a constant map.
Definition 3. Consider a function $f : I \to Y$, a Banach space. A continuous mapping $g : I \to Y$ is called an antiderivative of $f$ in $I$ if there exists a countable set $D \subset I$ such that for any $x \in I \setminus D$, $g$ is differentiable at $x$ and $g'(x) = f(x)$.

Proposition 10. If $g_1$ and $g_2$ are two antiderivatives of $f$ in $I$, then $g_1 - g_2$ is constant in $I$.

Proposition 11. Let $I \subset \mathbb{R}$ be the union of an increasing sequence of compact intervals $J_n$, $I = \bigcup_{n \in \mathbb{N}} J_n$. If $g_n$ is an antiderivative in $J_n$ of $f$ restricted to $J_n$, then the function $g : I \to X$ given by $g|_{J_n} = g_n$ is an antiderivative of $f$ on $I$.

Proof. Let $x_0$ be an interior point of $J_1 \subset J_n$ for all $n \in \mathbb{N}$, since the $J$’s are increasing. For each $n$, let $g_n$ be the antiderivative in $J_n$ of the restriction of $f$ to $J_n$ such that $g_n(x_0) = 0$, which is uniquely determined by the previous proposition. Then the restriction of $g_{n+1}$ to $J_n$ is an antiderivative of $f$ in $J_n$ such that $g_{n+1}(x_0) = 0$. Once again by the uniqueness proposition, $g_{n+1}|_{J_n} = g_n$. We can, therefore, define the mapping $g : I \to X$ as equal to $g_n$ in each of the $J_n$, which is an antiderivative of $f$ in $I$.

Definition 4. Let $I \subset \mathbb{R}$ be some interval and $F$ be a Banach space. We say that $f : I \to F$ is a step-function if there is an increasing finite sequence $\{x_i\}_{0 \leq i \leq n}$ of points of $\bar{I}$ such that $x_0 = a$ and $x_n = b$, where $a$ and $b$ are the origin and extremity respectively, and $f$ is constant in each of the open intervals $(x_i, x_{i+1})$ for $0 \leq i \leq n-1$.  

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Definition 5. Let $I = [a, b] \subset \mathbb{R}$ be a compact interval and $F$ a Banach space. Then $f : I \to F$ is called a regulated mapping if

1. $\forall x \in I \lim_{y \to x^+} f(y)$ and $\lim_{y \to x^-} f(y)$ exist

2. $f$ is the uniform limit of step functions

Theorem 9. Given $I$ an open subset of $\mathbb{R}$ and $X$ a Banach space, and a function $f : I \to X$ a regulated mapping. Then $f$ has an antiderivative in $I$.

Proof. By Proposition [11] we can assume that $I$ is compact, $I = [a, b]$. This follows from the fact that any $I$ can be decomposed into the union of increasing and compact $J_n$. We begin our proof by considering step functions.

Let $f$ be a step function and $(x_i)$ for $i = 0, \ldots, n - 1$ be an increasing sequence of points in $I = [a, b]$ such that

$$f(x) = c_i \text{ for } x \in (\lambda_i, \lambda_{i+1})$$

where $a = \lambda_0 < \lambda_1 < \cdots < \lambda_n = b$. We define the function $g : I \to X$ such that in each interval $[\lambda_i, \lambda_{i+1}]$, we have

$$g(x) = c_i(x - \lambda_i) + \sum_{k=0}^{i-1} c_k(\lambda_{k+1} - \lambda_k), \quad i = 0, \ldots, n - 1.$$
Given the above definition, we note that

$$\lim_{x \to \lambda_{i+1}} g(x) = c_i(\lambda_{i+1} - \lambda_i) + \sum_{k=0}^{i-1} c_k(\lambda_{k+1} - \lambda_k) = \sum_{k=0}^{i} c_k(\lambda_{k+1} - \lambda_k) = g(\lambda_{i+1}),$$

where \(i = 0, \ldots, n - 1\). Thus, \(g\) is well-defined and has derivative \(f\).

If an arbitrary regulated function \(f\) maps a compact interval into \(X\), \(f : [a, b] \to X\), then \(f\) is the limit of a uniformly convergent sequence of step functions, \(f_n\). We just showed that each step function \(f_n\) has an antiderivative \(g_n\). Therefore, we have that \(g = \lim_{n \to \infty} g_n\) is an antiderivative of \(f\).

The importance of the above is that if \(f\) is a regulated function, it has an antiderivative. If \(f\) has an antiderivative, we can define integration.

**Definition 6.** If \(g : I \to X\) is any antiderivative of a regulated function \(f : I \to X\), the difference \(g(b) - g(a)\) for any two points of \(I\) is called the **integral of** \(f\) **between** \(a\) and \(b\). It is written as

$$\int_a^b f(x) \, dx = g(b) - g(a).$$

For convenience, if \(g\) is an antiderivative of a regulated function \(f\), we write \(g'\) instead of \(f\), even though \(g\) may not be differentiable everywhere and even when the derivative exists, it may fail to be equal to \(f\).

**Proposition 12** (Linearity of Integration). Let \(f : I \to X\) be a regulated mapping.
and let $c$ be any continuous linear mapping of $X$ to $Y$, $c \in \mathcal{L}(X,Y)$. Then

$$\int_a^b c(f(x)) \, dx = c \left( \int_a^b f(x) \, dx \right). \quad (A.4)$$

**Proof.** Let $g : I \to X$ be an antiderivative of $f$. Let $u$ be a continuous linear map, $u : X \to Y$. We know that the derivative of a linear map is linear. Using the Chain Rule and the fact that $Du(x_0) = u$, we write

$$(u \circ g)'(x) = u'(g(x)) \circ g'(x) = u \circ f = u(f(x)).$$

This tells us that $u \circ g$ is an antiderivative of $u(f(x))$. Thus, we may rewrite the left-hand side of (A.4) as

$$\int_a^b u(f(x)) \, dx = u(g(b)) - u(g(a)).$$

However, $u$ is a linear function. Therefore, we may write the right-hand side of (A.4) as

$$u \left( \int_a^b f(x) \, dx \right) = u(g(b) - g(a)) = u(g(b)) - u(g(a)).$$

□

**Lemma 35.** Let $E$ be a compact metric space, $F$ and $G$ be metric spaces and $\Omega \subset F$ be a compact subset. Furthermore, let $u : E \times F \to G$ be continuous. Then there is an open neighborhood $\Lambda \subset \Omega$ such that $u : E \times \Lambda \to G$ is bounded.
Proof. First we would like to establish the fact that for any \( e \in E \) and \( \omega \in \Omega \) there is an open ball \( Q_{e,\omega} \subset E \) and an open ball \( R_{e,\omega} \subset F \) such that \( u(Q_{e,\omega} \times R_{e,\omega}) \) is bounded since \( u \) is continuous. So let’s take a bounded set \( B \subset G \) with \( u(e,\omega) \in B \). Then we will have \( u^{-1}(B) \subset E \times F \) be an open subset. Therefore, we can find open balls \( Q_{e,\omega} \subset E \), \( R_{e,\omega} \subset F \) with \( Q_{e,\omega} \times R_{e,\omega} \subset u^{-1}(B) \). Thus, by construction, \( u(Q_{e,\omega} \times R_{e,\omega}) \) is bounded. Now for each \( \omega \in \Omega \) we have that the collection \( \{Q_{e,\omega} | e \in E\} \) is an open covering for \( E \). Since \( E \) is compact, we can find a finite subcovering \( \{Q_{e_j,\omega} | j = 1,\ldots,n\} \). Now choose \( V_e \) to be the \( R_{e_j,\omega} \) with smallest radius. Then \( u(E \times V_e) \) is bounded. Now we can observe that \( \{V_e | e \in E\} \) is an open covering for \( \Omega \). Since \( \Omega \) is compact we have a finite subcovering \( \{V_{e_k} | k = 1,\ldots,n\} \). By defining \( \Lambda = \bigcup V_{e_k} \) we have that \( u : E \times \Lambda \to G \) is bounded. \(\)

**Definition 7.** Let \((X,d)\) be a metric space and \( f : X \to X \). \( f \) is called a contraction mapping if there exists \( 0 \leq c < 1 \) with \( d(f(x), f(y)) \leq cd(x,y) \) for all \( x,y \in X \).

**Theorem 10** (Banach Contraction Theorem). Let \( X \) be a complete metric space and \( f : X \to X \) be a contraction mapping. Then \( f \) has a unique fixed point in \( X \).

**Theorem 11** (Fixed Point Theorem). Let \( F \) be a Banach space, \( B(y_0, \beta) \subset F \) and \( v : B(y_0, \beta) \to F \) such that \( \|v(y_1) - v(y_2)\|_F \leq k \cdot \|y_1 - y_2\|_F \) for any pair \( y_1, y_2 \in B(y_0, \beta) \) and \( 0 \leq k < 1 \). Furthermore, if \( \|v(y_0) - y_0\|_F \leq k(1-k) \) then there exists a unique fixed point \( z \in B(y_0, \beta) \) such that \( v(z) = z \).

**Proof.** Define \( f : B(0, \beta) \to B(0, \beta) \) by \( f(x) = v(x+y_0) - y_0 \). Then we have that \( f \) is a contraction mapping on \( B(0, \beta) \). So by the Banach Contraction Mapping Theorem
10 $f$ will have a unique fixed point $x_0$. Then it follows that $x_0 + y_0 = v(x_0 + y_0)$. This completes the proof.

**Theorem 12.** (The Fundamental Theorem of ODE) Let $Y$ be a Banach space, $X \subset Y$ an open subset, $I \subset \mathbb{R}$, and $f : I \times X \to Y$ a continuously differentiable map. Then for any $t_0 \in I$ and $x_0 \in X$ there exists an open ball $J \subset I$ and a unique differentiable mapping $u : J \times X \to Y$ such that for all $t \in J$,

$$u'(t, x) = f(t, u(t, x)) \quad \text{and} \quad u(t_0, x_0) = x_0.$$  

**Proof.** By Lemma 35 there are radii $r_1, r_2 > 0$ such that for $B_1 = B(t_0, r_1)$ and $B_2 = B(x_0, r_2)$, we have

$$M = \sup_{(t,x) \in B_1 \times B_2} \| f(t, x) \|_Y < \infty \quad \text{and} \quad K = \sup_{(t,x) \in B_1 \times B_2} \| Df(t, x) \|_Y < \infty.$$  

Now, for $r < \min(r_1, \frac{r_2}{M + Kr_2})$ let’s define the following quantities on $J_r = B(t_0, r)$:

$$F_r = \{ y : J_r \to Y \},$$  

$$\| y \|_{F_r} = \sup_{t \in J_r} \| y(t) \|_Y,$$  

$$C_{x_0} : J_r \to Y \quad \text{by} \quad C_{x_0}(t) = x_0,$$  

$$V_r = B(C_{x_0}, r) \subset F_r.$$  

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By Theorem 9 we can define the map $T : V_r \to F_r$ given by

$$Ty(t) = x_0 + \int_{t_0}^{t} f(s, y(s))ds.$$ 

Now we must check that $T$ satisfies the Fixed Point Theorem 11 since the fixed point $y(t)$ will be a solution to our ODE. So we will proceed by investigating the computations of $||Ty_1 - Ty_2||_{F_r}$, for $y_1, y_2 \in F_r$, and $||TC_{x_0} - C_{x_0}||_{F_r}$. It follows that

$$||Ty_1 - Ty_2||_{F_r} = \sup_{t \in J_r} \left( ||x_0 + \int_{t_0}^{t} f(s, y_1(s))ds - \left( x_0 + \int_{t_0}^{t} f(s, y_2(s))ds \right) ||_Y \right)$$

$$= \sup_{t \in J_r} \left( \int_{t_0}^{t} (f(s, y_1(s)) - f(s, y_2(s)))ds \right) ||_Y$$

$$\leq \sup_{t \in J_r} \int_{t_0}^{t} \|f(s, y_1(s)) - f(s, y_2(s))\|_Y ds. \quad (A.5)$$

By the Mean Value Theorem 8 we have

$$\|f(s, y_1(s)) - f(s, y_2(s))\|_Y \leq K\|y_1(s) - y_2(s)\|_Y.$$
Substituting this into (A.5) yields

\[ \| Ty_1 - Ty_2 \|_{F_r} \leq \sup_{t \in J_r} \int_{t_0}^{t} K \| y_1(s) - y_2(s) \|_Y ds \]

\[ \leq \sup_{s \in J_r} \| y_1(s) - y_2(s) \|_Y \cdot K \cdot \sup_{t \in J_r} \int_{t_0}^{t} ds \]

\[ = \sup_{t \in J_r} (t - t_0) \cdot K \cdot \| y_1 - y_2 \|_{F_r} \]

\[ \leq K r \| y_1 - y_2 \|_{F_r}. \]

By assumption we have \( Kr = \frac{Kr_2}{M + Kr_2} < 1 \) which implies that \( T \) satisfies the first hypothesis of the Fixed Point Theorem. Now we observe that

\[ \| TC_{x_0} - C_{x_0} \|_{F_r} = \sup_{t \in J_r} \| x_0 + \int_{t_0}^{t} f(s, C_{x_0}(s)) ds - x_0 \|_Y \]

\[ = \sup_{t \in J_r} \int_{t_0}^{t} f(s, C_{x_0}(s)) ds \|_Y \]

\[ \leq \sup_{t \in J_r} \int_{t_0}^{t} \| f(s, C_{x_0}(s)) \|_Y ds \]

\[ \leq \sup_{(s, x_0) \in B_1 \times B_2} \| f(s, C_{x_0}(s)) \|_Y \cdot \sup_{t \in J_r} (t - t_0) \]

\[ \leq Mr \]

\[ < \frac{Mr_2}{M + Kr_2} \]

\[ = r_2 \left( 1 - \frac{Kr_2}{M + Kr_2} \right). \]
This implies that $T$ satisfies the second hypothesis of the *Fixed Point Theorem* which completes our proof.
APPENDIX B

TOP TEN THEOREMS AND LEMMAS

**Theorem 13** (Cauchy-Schwarz Inequality). Let \( f, g \in L^2(\mathbb{T}) \)

\[
\left| \int_{\mathbb{T}} f(x)g(x) \, dx \right| \leq \|f\|_{L^2(\mathbb{T})} \|g\|_{L^2(\mathbb{T})}.
\]

**Theorem 14** (Sobolev Embedding Theorem). For \( k \in \mathbb{N} \) and \( s > k + \frac{1}{2} \), the space \( H^s \) embeds continuously into \( C^k(\mathbb{T}) \). In particular, we have the inequality

\[
\|f\|_{C^k} \leq c_s \|f\|_{H^s}.
\]

**Theorem 15** (Algebra Property). For \( u, v \in H^s \) and \( s > \frac{1}{2} \), we have

\[
\|u \cdot v\|_{H^s} \lesssim \|u\|_{H^s} \|v\|_{H^s}.
\]

**Theorem 16** (Rellich’s Theorem). For any compact \( M \) and \( s \in \mathbb{R} \), the inclusion operator \( j : H^{s+\sigma}(M) \hookrightarrow H^s(M) \) is compact for any \( \sigma > 0 \).
Theorem 17 ( Alaoglu’s Theorem). If $X$ is a normed vector space, the closed unit ball $B^* = \{ f \in X^* : \| f \| \leq 1 \}$ in $X^*$ is compact in the weak* topology.

Theorem 18 (Ascoli’s Theorem). Let $X$ be a Banach space and $M$ a compact metric space. If $S \subset C(M, X)$ such that

1. $S$ is equicontinuous; i.e. $\sup \{ \| f \| : f \in S \} < \infty$ and
2. for each $x \in M$ the set $S(x) = \{ f(x) \}$ is precompact in $X$; i.e. the closure of $S(x)$ is compact,

then $S$ is precompact in $C(M, X)$.

Lemma 36. (Kato-Ponce) If $s > 0$ then there is a $c_s > 0$ such that

$$\| [D^s, f] g \|_{L^2} \leq c_s \left( \| D^s f \|_{L^2} \| g \|_{L^\infty} + \| \partial_x f \|_{L^\infty} \| D^{s-1} g \|_{L^2} \right).$$

(B.1)

Lemma 37 (Interpolation Lemma). Let $f \in H^s$ and $s_1 < s < s_2$. Then

$$\| f \|_{H^{s'}} \leq \| f \|_{H^{s_1}}^{\frac{s_2-s}{s_2-s_1}} \| f \|_{H^{s_2}}^{\frac{s-s_1}{s_2-s_1}}.$$

Lemma 38 (Hölder’s Inequality). Let $1 \leq p$ and $q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} \leq 1$. If $f \in L^p$ and $g \in L^q$, then for $fg \in L^r$, we have

$$\| fg \|_{L^r} \leq \| f \|_{L^p} \| g \|_{L^q}.$$
Lemma 39. For $v \in H^s$,

$$\| \partial_x J_\epsilon v \|_{H^s} \leq \alpha_\epsilon \| v \|_{H^s}$$

for $\alpha_\epsilon \in \mathbb{R}$.

Proof of Lemma 37:

$$\| \partial_x J_\epsilon v \|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{\partial_x J_\epsilon v}(\xi)|^2$$

Since $\partial_x J_\epsilon v = \partial_x (j_\epsilon * v) = (\partial_x j_\epsilon) * v$, we have

$$\| \partial_x J_\epsilon v \|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |(\partial_x j_\epsilon) * v(\xi)|^2 \leq \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{\partial_x j_\epsilon}(\xi)|^2 |\hat{v}(\xi)|^2.$$

Replacing the equality $\hat{\partial_x j_\epsilon}(\xi) = -i\xi \hat{j_\epsilon}(\xi)$ yields

$$\| \partial_x J_\epsilon v \|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |(\partial_x j_\epsilon) * v(\xi)|^2 \leq \int_{\mathbb{R}} (1 + \xi^2)^s |\partial_x j_\epsilon(\xi)|^2 |\hat{v}(\xi)|^2.$$

Since $j_\epsilon(\xi) = j(\epsilon \xi)$, we have

$$\| \partial_x J_\epsilon v \|_{H^s}^2 = \int_{\mathbb{R}} (1 + \xi^2)^s |j_\epsilon(\epsilon \xi)|^2 |\hat{v}(\xi)|^2.$$
Given that $j \in \mathbb{N}$, $\hat{j} \in \mathbb{T}$. In particular, this means that for $\xi \in \mathbb{R}$ and $\varepsilon \in (0, 1]$, $|\hat{j}(\varepsilon \xi)|_{0,1} = |\varepsilon \hat{\xi}(\varepsilon \xi)| \leq \alpha$ for some constant $\alpha$. Therefore, $|\xi \hat{j}(\varepsilon \xi)| \leq \alpha / \varepsilon = \alpha_\varepsilon$, so

\[
\|\partial_x J_\varepsilon v\|_{H^s}^2 \leq \int_{\mathbb{R}} (1 + \xi^2)^s \alpha_\varepsilon^2 |\hat{v}(\xi)|^2.
\]

This then yields the estimate

\[
\|\partial_x J_\varepsilon v\|_{H^s} \leq \alpha_\varepsilon \|v\|_{H^s}.
\]
APPENDIX C

OTHER USED THEOREMS AND LEMMAS

Lemma 40. If $f \in H^s$, then $\|J_\varepsilon f\|_{H^s} \leq \|f\|_{H^s}$.

Proof.

$$\|J_\varepsilon f\|_{H^s}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s \left| \hat{J}_\varepsilon f \right|^2$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s \left| \hat{j}_\varepsilon * f \right|^2$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s \left| \hat{j}_\varepsilon \hat{f} \right|^2$$

We use that $\hat{j}_\varepsilon(k) \in [0, 1]$ to finish the proof.

$$\|J_\varepsilon f\|_{H^s}^2 \leq \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}|^2$$

$$= \|f\|_{H^s}^2.$$  

In particular, we now know that $J_\varepsilon$ is continuous from $H^s$ to $H^s$.  

$\square$
Remark: Analogous result holds for the real line by just replacing the sum with the integral.

Lemma 41. $D^s$ and $J_\varepsilon$ commute.

Proof.

\[
\hat{J}_\varepsilon D^s f(k) = \hat{j}_\varepsilon \ast D^s f(k) \\
= \hat{j}_\varepsilon(k)D^s f(k) \\
= \hat{j}_\varepsilon(k)(1 + k^2)^{s/2} \hat{f}(k) \\
= (1 + k^2)^{s/2} \hat{j}_\varepsilon(k) \hat{f}(k) \\
= (1 + k^2)^{s/2} \hat{j}_\varepsilon \ast f(k) \\
= (1 + k^2)^{s/2} \hat{J}_\varepsilon f(k) \\
= D^s J_\varepsilon f(k)
\]

which shows us that

\[
J_\varepsilon D^s f = D^s J_\varepsilon f.
\]

Theorem 19 (Parseval’s Formula). If $f, g \in L^2$, then

\[
\int_{\mathbb{R}} \hat{f} \hat{g} d\xi = \int f(x)g(x) dx.
\]
Lemma 42. $J_\varepsilon$ is a self-adjoint operator; i.e.,

$$\int (J_\varepsilon f)g \, dx = \int f (J_\varepsilon g) \, dx$$

Proof.

$$\int (J_\varepsilon f)g \, dx = \int \hat{J_\varepsilon f} \hat{g} \, dx \text{ by Parseval’s Formula} \quad [19]$$

$$= \int \hat{j_\varepsilon} \hat{f} \hat{g} \, dx$$

$$= \int \hat{f} \hat{j_\varepsilon^*} \hat{g} \, dx$$

$$= \int \hat{f} \hat{j_\varepsilon} \hat{g} \, dx$$

$$= \int \hat{j_\varepsilon} \hat{f} \hat{g} \, dx$$

$$= \int \hat{f} \hat{J_\varepsilon} g \, dx$$

$$= \int f J_\varepsilon g \, dx \text{ by another application of Parseval’s Formula} \quad [19]$$

Lemma 43. Let $f \in H^s$ for any $s > 3/2$. Then

$$\|\partial_x f\|_{H^s} \leq \|f\|_{H^{s+1}}.$$  

Proof. Since $s > 3/2$, the Sobolev Embedding Theorem tells us that any element of
$H^s$ is also in $C^1$.

$$\|\partial_x f\|_{H^s}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{\partial_x f}(k)|^2$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s k^2 |\hat{f}(k)|^2$$

$$\leq \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s k^2 |\hat{f}(k)|^2$$

$$\leq \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s (1 + k^2) |\hat{f}(k)|^2$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}(k)|^2$$

$$= \| f \|_{H^{s+1}}^2.$$  

\[\square\]

**Lemma 44.** $\| f \|_{H^{s-1}} \leq \| f \|_{H^s}$

**Proof.** This is a property of Sobolev.

$$\| f \|_{H^{s-1}}^2 = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^{s-1} |\hat{f}(\xi)|^2$$

$$\leq \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + \xi^2)^{s-1} (1 + k^2) |\hat{f}(\xi)|^2$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}|^2$$

$$= \| f \|_{H^s}^2.$$  

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Lemma 45. Given \( f \in H^s \),
\[
D^s \partial_x f = \partial_x D^s f.
\]

Proof.
\[
\begin{align*}
\hat{D^s \partial_x f}(k) &= (1 + k^2)^{s/2} \hat{\partial_x f}(k) \\
&= (1 + k^2)^{s/2} (-ik) \hat{f}(k) \\
&= -ik(1 + k^2)^{s/2} \hat{f}(k) \\
&= -ik \hat{D^s f}(k) \\
&= \hat{\partial_x D^s f}(k)
\end{align*}
\]

Lemma 46. Given \( f \in H^s \), we have the following equality
\[
\|D^s f\|_{L^2} = \|f\|_{H^s}.
\]

Proof.
\[
\begin{align*}
\|f\|_{H^s}^2 &= \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |\hat{f}(k)|^2 \\
&= \sum_{k \in \mathbb{Z}} |\hat{D^s f}(k)|^2 \\
&= \|D^s f\|_{L^2}^2.
\end{align*}
\]

\( \Box \)

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Theorem 20. Let $A$ be an open, connected subset of a Banach space $X$ and $f_n : A \to Y$ be a sequence of differentiable functions from $A$ into the Banach space $Y$. Suppose that

1. There is an $x_0 \in X$ such that the sequence $\{f_n(x_0)\}$ converges in $Y$.

2. For every $a \in A$, there is a ball $B(a) \subset A$ centered at $a$ such that $\{f'_n\}$ converges uniformly in $B(a)$.

Then for each $a \in A$, the sequence $\{f_n\}$ converges uniformly in $B(a)$. Moreover, if for each $x \in A$

$$f(x) = \lim_{n \to \infty} f_n(x) \text{ and } g(x) = \lim_{n \to \infty} f'_n(x)$$

then $g(x) = f'(x)$ for every $x \in A$.

Lemma 47. Given $f \in H^s$ and $g \in H^{s+1}$, we have

$$|\langle f, g \rangle_{H^s}| \leq \|f\|_{H^{s-1}} \|g\|_{H^{s+1}}.$$ 

Proof.

$$|\langle f, g \rangle_{H^s}|^2 = \left| \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}(k)|^2 |\hat{g}(k)|^2 \right|$$

$$= \left| \sum_{k \in \mathbb{Z}} (1 + k^2)^{s+1} |\hat{f}(k)|^2 (1 + k^2)^{s+1} |\hat{g}(k)|^2 \right|.$$
By the Cauchy-Schwartz inequality we have

\[
|\langle f, g \rangle_{H^s}|^2 \leq \left| \sum_{k \in \mathbb{Z}} (1 + k^2)^{\frac{s+1}{2}} |\hat{f}(k)|^2 \right| \left| \sum_{k \in \mathbb{Z}} (1 + k^2)^{\frac{s+1}{2}} |\hat{g}(k)|^2 \right|
\]

\[
= \|f\|_{H^{s+1}}^2 \|g\|_{H^{s+1}}^2.
\]

\[\square\]

**Lemma 48.** Let \( f \in H^{s+1} \). Then

\[
\|f\|_{H^{s+1}} \leq \|f\|_{H^s} + \|\partial_x f\|_{H^s}.
\]

**Proof.**

\[
\|f\|_{H^{s+1}}^2 = \sum_{k \in \mathbb{Z}} (1 + k^2)^{s+1} |\hat{f}(k)|^2
\]

\[
= \sum_{k \in \mathbb{Z}} (1 + k^2)^s (1 + k^2) |\hat{f}(k)|^2
\]

\[
= \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\hat{f}(k)|^2 + \sum_{k \in \mathbb{Z}} (1 + k^2)^s k^2 |\hat{f}(k)|^2
\]

\[
= \|f\|_{H^s}^2 + \sum_{k \in \mathbb{Z}} (1 + k^2)^s |\partial_x f(k)|^2
\]

\[
= \|f\|_{H^s}^2 + \|\partial_x f\|_{H^s}^2.
\]
Therefore, we have

\[ \|f\|_{H^{s+1}} = \left( \|f\|_{H^{s}}^2 + \|\partial_x f\|_{H^{s}}^2 \right)^{1/2} \leq \|f\|_{H^{s}} + \|\partial_x f\|_{H^{s}}. \]

Lemma 49. The function \(j_\varepsilon\) satisfies the property

\[ \hat{j}_\varepsilon(k) = \hat{j}(\varepsilon k). \]

Proof. We use the definition of \(j_\varepsilon\) to write

\[
\begin{align*}
\hat{j}_\varepsilon(k) &= \int_T e^{-ikx} j_\varepsilon(x) \, dx \\
&= \int_T e^{-ikx} \left( \frac{1}{2\pi} \sum_{m\in\mathbb{Z}} e^{imx} \hat{j}(\varepsilon m) \right) \, dx \\
&= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \hat{j}(\varepsilon m) \int_T e^{ix(m-k)} \, dx.
\end{align*}
\]

Since \(\{e^{ixn}\}_{m\in\mathbb{Z}}\) is orthogonal in \(l^2(\mathbb{T})\), we have

\[ \hat{j}_\varepsilon(k) = \frac{1}{2\pi} \hat{j}(\varepsilon m)(2\pi) = \hat{j}(\varepsilon k). \]

Remark: By switching the summation with the integral, one may find analo-
gous results for the case of the real line.
APPENDIX D

STRONG, WEAK AND WEAK* CONVERGENCE

Let $X$ be a normed vector space and let $\{x_n\}$ be a sequence in $X$. We say that $x_n \to x$ strongly if

$$\|x_n - x\|_X \to 0.$$  \hfill (D.1)

In other words, we have that strong convergence is identified as convergence in norm.

We have that the weak topology on $X$ is generated by $X^*$, the dual of $X$, and the convergence with respect to this topology is known as weak convergence. Let $(x_n)$ be a net in $X$. We say that $x_n \to x$ converges weakly in $X$ if for all $f \in X^*$ we have that

$$f(x_n) \to f(x).$$  \hfill (D.2)

Here we shall take a moment to inform the reader of directed sets and nets. A directed set is a set $A$ equipped with a binary relation $\preceq$ such that

- $\alpha \preceq \alpha$;

- if $\alpha \preceq \beta$ and $\beta \preceq \sigma$ then $\alpha \preceq \sigma$;
• for any $\alpha, \beta \in A$ there exists $\sigma \in A$ such that $\alpha \preceq \sigma$ and $\beta \preceq \sigma$.

A net in $X$ is a mapping $\alpha \to x_\alpha$ from a directed set $A$ into $X$. We denote this mapping as $(x_\alpha)$. Now from the previous analysis on the aforementioned topologies, we say that the original topology on $X$ is the strong topology on $X$ while the weak topology on $X$ is generated by $X^*$ which we call the dual of $X$.

Now, we have that the weak topology on $X^*$ is generated by $X^{**}$ (the one of more interest is the topology generated by $X$ which is considered a subspace of $X^{**}$) which is called the weak* topology and the convergence with respect to this topology is known as weak* convergence. We say that a sequence $\{\varphi_n\}$ in $X^*$ converges to $\varphi$ weak*-ly if for all $x \in X$ we have that

$$\varphi_n(x) \to \varphi(x).$$

(D.3)

Note: The way $X$ is considered a subspace of $X^{**}$ is through the canonical inclusion map $\Phi_\varphi(x) = \varphi(x)$. One can see that this map is continuous by

$$|\Phi_\varphi(x)| = |\varphi(x)| \leq \|\varphi\||x| \leq \|x\| < \infty.$$

To recap the above we have:

1. Strong Convergence: $\{x_n\} \subset X$ and $x_n \to^s x$ if $\|x_n - x\|_X \to 0$

2. Weak Convergence: $(x_n) \subset X$ is a net and $x_n \to^w x$ if $\forall f \in X^*$ we have $f(x_n) \to f(x)$
3. **Weak* Convergence:** \( \{ \varphi_n \} \subset X^* \) and \( \varphi_n \rightharpoonup^{w^*} \varphi \) if \( \forall x \in X \), \( \varphi_n(x) \to \varphi(x) \).

Furthermore, we note that the weak and weak* topologies coincide when \( X \) is reflexive, or \( X^{**} = X \). Since we deal with Sobolev spaces \( H^s \), we know the weak and weak* topologies to coincide since \( H^s \) is a Hilbert space and all Hilbert spaces are reflexive.

**Examples of Strong Convergence:**

1. Let \( f_n(x) = \chi_{[0,1]}(x) \cdot \frac{1}{n} \in L^2(\mathbb{R}) \). Then we have that

\[
\int_{\mathbb{R}} |f_n(x)|^2 \, dx = \int_{0}^{1} \frac{1}{n^2} \, dx = \frac{1}{n^2} \to 0, \quad n \to \infty.
\]

Thus, we have that \( f_n \to 0 \) in \( L^2(\mathbb{R}) \).

2. \( \frac{1}{2\pi} \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \to f(x) \) in \( L^2[0, 2\pi] \).

3. Let \( H \) be a Hilbert space and \( \{ e_n \}_{n=1}^{\infty} \) be an orthonormal basis. Then for any \( x \in H \)

\[
\sum_{n=-N}^{N} \langle x, e_n \rangle e_n \to x \quad \text{in} \quad H.
\]

4. For \( L^2(\mathbb{R}) \) we have that \( \{ e_n \} = \{ H_n(x)e^{-x^2/2} \} \), where \( H_n \) are Hermite polynomials, is an orthonormal basis. (Proof: See Reed & Simon’s Functional Analysis).

5. Let \( \varphi_\epsilon \) be mollifiers on \( \mathbb{R} \). Then we have that for any \( f \in H^s(\mathbb{R}) \),

\[
\| f - f * \varphi_\epsilon \|_{H^s} \to 0, \quad \text{as} \quad \epsilon \to 0.
\]
Examples of Weak and Weak* Convergence:

Let \( f_n \in L^2 \). Then if \( f_n \to f \) in \( L^2 \), we have that \( f_n \to f \) weakly in \( L^2 \). This follows from the fact that \( L^2 \) is a Hilbert space, thus reflexive and hence its own dual. So let \( \varphi \in (L^2)^* = L^2 \) and we have that
\[
(f_n, \varphi) - (f, \varphi) = \int_{\mathbb{R}} \varphi(f_n - f) dx \leq \|\varphi\|_{L^2} \|f_n - f\|_{L^2} \to 0 \quad \text{as} \quad n \to \infty.
\]

For another good example of weak convergence, we examine the Hilbert space \( L^2[0, 2\pi] \) which is the space of square-integrable functions on the interval \([0, 2\pi]\) equipped with the inner product defined by
\[
(f, g) = \int_{0}^{2\pi} f(x) \cdot g(x) dx
\]
for \( f, g \in L^2[0, 2\pi] \). Now, let the sequence of functions \( \{f_n\} \subset L^2[0, 2\pi] \) be defined by \( f_n = \sin(nx) \). We have that \( f_n \to^w 0 \) as the integral
\[
(f_n, g) = \int_{0}^{2\pi} \sin(nx) \cdot g(x) dx \to 0, \quad \text{i.e.} \quad (f_n, g) \to (0, g) = 0.
\]

To see this, we note that simple functions are dense in \( L^p \) for \( p \in [1, \infty) \). So we have that \( g(x) \) can be approximated via simple functions, or similarly
\[
g(x) = \sum_{1}^{n} a_j \chi_{I_j}
\]
where each \( \chi_{I_j} \) is a simple function with \( I_j \subset [0, 2\pi] \) and the \( a_j \)'s are constants. Then
for each $\chi_{I_j}$, we have by plugging $g(x)$ in terms of simple functions back into (0.5) that

$$\int_0^{2\pi} \sin(nx) a_j \chi_{I_j} \, dx = \int_{x_{j_1}}^{x_{j_2}} a_j \sin(nx) \, dx = -a_j \frac{\cos(nx)}{n} \bigg|_{x_{j_1}}^{x_{j_2}} \to 0 \text{ as } n \to \infty.$$

Here we have that $f_n$ will be zero more frequently as $n \to \infty$, however, this does not mean it is similar to the zero function at all. This is why the convergence is said to be weak. See Figure D.1.

![Figure D.1. Sine waves of varied frequency.](image)

In fact, we have $\| \sin(nx) - 0 \|_{L^2}^2 = \pi$. Thus, we do not have strong convergence.

Now we give an example of weak* convergence, but not weak convergence.
Consider the sequence space $l^1$ which is regarded as the dual of the space $c_0$, which is the space of all sequences whose limit is zero. Let $e_n$ be the element with a 1 in the $n$’th spot and 0 elsewhere as given by

$$e_n = (0, \ldots, 0, 1, 0, \ldots).$$

Then we have that $e_n \to^{w^*} 0$ as a sequence of elements of $c_0^*$, since for any $a = (a_m) \in c_0$, we have

$$e_n(a) = a_n \to 0, \quad \text{as } n \to \infty.$$

It is not true, however, that $e_n \to^w 0$, because the dual space of $l^1$, which is $l^\infty$, contains the element

$$b = (1, 1, \ldots)$$

and $b(e_n) = 1$ for all $n$. Thus, we have weak* convergence, but not weak convergence.
APPENDIX E

ALAOGLU’S THEOREM

Theorem 21. (Alaoglu’s Theorem) If $X$ is a normed vector space, the closed unit ball $B^* = \{ f \in X^* : \| f \| \leq 1 \}$ in $X^*$ is compact in the weak* topology.

Proof: We have to show the following: given a bounded sequence in $X^*$, there is a weakly convergent subsequence.

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $X^*$. Let $\{x_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $X$. Choose subsequences $\mathbb{N} \supset \Lambda_1 \supset \Lambda_2 \cdots$ such that for every $j \in \mathbb{N}$ we have

$\varphi_k(x_j) \to a_j =: \varphi(x_j)$ as $k \to \infty$, $k \in \Lambda_j$.

Let $\Lambda$ be the diagonal sequence of $\Lambda_j$’s.

Claim 1: $\varphi$ can be extended to an element of $X^*$.

Proof:

$\varphi$ can be extended in a general way to a linear function on $M = \text{span}\{x_j\}_{j \in \mathbb{N}}$
by noticing that

$$\varphi(\text{span}\{x_j\}_{j \in \mathbb{N}}) = \varphi \left( \sum_{k=1}^{\infty} x_k \right) = \sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \varphi(x_k).$$

We extend \( \varphi \) to a functional in \( X^* \) by pointwise limit (notice that \( M \) is dense in \( X \)). We have

$$|\varphi(x)| = \lim_{k \to \infty} |\varphi(x_k)| \leq \limsup_{k \to \infty} \|\varphi_k\|_{X^*} \|x\|_X$$

where \( x_k \to x \) as \( k \to \infty \). Since \( \{\varphi_k\}_{k \in \mathbb{N}} \) was bounded, \( \varphi \) is bounded and thus continuous. □

Claim 2: \( \varphi_k \to w^* \varphi \) as \( k \to \infty, k \in \Lambda \).

Proof:

Let \( \lim_{j \to \infty, j \in J} x_j = x \in X \), where \( J \) is some subset of \( \mathbb{N} \). We now have

$$|\varphi_k(x) - \varphi(x)| \leq |\varphi_k(x - x_j)| + |\varphi(x - x_j)| + |\varphi_k(x_j) - \varphi(x_j)|$$

$$\leq (\sup_{i \in \Lambda} \|\varphi_i\|_{X^*} + \|\varphi\|_{X^*}) \|x - x_j\|_X + |\varphi_k(x_j) - \varphi(x_j)| \to 0$$

as both \( k \) and \( j \) tend towards infinity. □

This concludes the proof of our theorem. □
BIBLIOGRAPHY


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