DESIGN PROBLEMS IN DISTRIBUTED CONTROL OF MULTI-AGENT SYSTEMS

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Jie Liu

Vijay Gupta, Director

Graduate Program in Electrical Engineering
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Abstract

by

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Distributed control of multi-agent systems is known to be a difficult problem. In this dissertation, we study two challenges that make the problem difficult: (i) stabilizing the multi-agent system with limited information sharing among the agents and (ii) incentivizing the actions of the agents to align with the desired team objective.

In the first set of problems, we consider a large scale system in which various components such as sensors, controllers and actuators are interconnected by communication networks. It is known that the stabilization of such a distributed system is affected by the limits imposed by the communication channels such as data erasures, limited data rate and delays, etc. We begin by considering the data erasure effect of the communication network. By combining tools from control theory, network theory and information theory, we provide a systematic study of both explicit and implicit information flow networks in the system and provide necessary and sufficient stabilizability conditions. Then, we model the communication network as a Gaussian multi-user channel and obtain the rate regions of stabilizability for two systems controlled over Gaussian Multiple Access channels (MAC). In the second half of the dissertation, we study the design of markets for various agents in a multi-agent system to incentivize their actions to be aligned with the social planner’s objective. For concreteness, we focus on the design of markets for plug-in electric vehicle charging
infrastructure. First, we set up the problem of locations and price optimization for commercial charging stations. We model the problem as a static price competition between competing charging stations with the customers deciding among the stations based on their own utility function. A two-level hierarchical game is formulated to study the selfish routing of the customers as a lower-level congestion game and the pricing games between stations as the upper-level game. We characterize the existence and properties of the static equilibrium solution. The second problem addresses the pricing of electric vehicle charging trajectories by the charging station owner when some electric vehicles can offer the added flexibility of temporary discharge. We obtain the competitive equilibrium solution of the charging rate profiles and service price profile. This equilibrium in the dynamic setting is later extended to consider a scenario closer to Stackelberg game. Social welfare of each equilibrium is also evaluated.
To Meng Xia, my dearest wife.
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CHAPTER 1

INTRODUCTION

1.1 Background

Large scale distributed systems in which a large number of components interact across communication network are central to modern life. A systematic analysis and design theory for such systems is, however, still lacking. In spite of the many recent advances in this direction, fundamental issues and challenges remain. For one, limited information exchange across the communication networks makes the control of such systems, even through a central control planner, fundamentally difficult. Traditional control theory does not consider the effect of decentralization or communication channels very well and basic results such as separation quickly break down in this scenario. Further, when the planners must design economic incentives for various components to align their actions for systems performance, the problem remains open in general.

In this dissertation, we consider some problems in this area. The first problem is that of distributed stabilization of a large scale system when various components interact across communication networks. Without the communication networks, this problem has traditionally been studied in distributed control. The central finding there is that information exchange among various controllers through driving the plant state to certain regions — also called signaling — is crucial. With only one controller but communication networks being present, the problem is studied in the area of networked control systems. The central idea in this direction is that design of control oriented encoders and decoders is crucial. Our work combines both these
perspectives. Using tools from information theory, network theory, and control, we provide necessary and sufficient condition for stabilization of distributed system over two types of communication networks.

The second problem we consider is the design of markets and economic incentives for various agents to align their goals with the planner’s goal. For concreteness, we consider the design of plug-in electric vehicle (PEV) charging infrastructures. Unlike the plug-in hybrid electric vehicle (PHEV), PEVs rely solely on electrical power provided by an on-board battery, without consuming any fossil fuel. As the penetration level of PEVs increases, there will be a heavy burden on the existing power transmission and distribution networks. We consider two problems in charging station location and pricing. In the first problem, we show how competing stations can set their prices when the customers’ utility function also includes waiting time and distance driven to access the charging station. In the second problem, we show how the charging station can compensate customers for providing extra flexibility by allowing the stations to discharge these batteries. In both these problems, techniques from game theory must be altered to allow for technological considerations — for instance, decrease in the utility of a customer due to degraded battery life from discharge. We once again provide a framework to address these issues.

1.2 Outline

In this section, we provide an outline of the dissertation and briefly introduce the topics of the remaining chapters.

The first problem, addressed in Chapter 2, is to stabilize a decentralized system with multiple control stations. We consider analog erasure channels connecting the sensors, controllers and actuators. We wish to identify conditions for stabilizability of such a system. A necessary condition for stabilizability of a decentralized control system is obtained. By constructing a specific algorithm, we also prove that this
condition is sufficient for stabilizability.

The second problem, addressed in Chapter 3, is to stabilize two scalar linear time invariant (LTI) systems across a Gaussian multiple access channel (MAC) in the mean square sense. In this system, the two states are encoded and sent separately across a Gaussian MAC channel and received by a joint decoder. The channel output is the sum of the two inputs and an additive Gaussian white noise. In this problem, we first assume that the two scalar plants can be stabilized in a decentralized manner, i.e. the encoders do not exchange information with each other. Under this assumption, a necessary condition is obtained, which shows that the log sum of unstable eigenvalues of both systems must be within the same capacity region of feedback Gaussian MAC channel. For sufficiency, we design a time-division control law and communication scheme and achieve new points in the stabilizability region, compared to the existing work [83]. When the encoders are allowed to exchange information, a condition that is both necessary and sufficient is obtained.

In Chapter 4, we consider the problem of a static price competition between charging stations located in different geographic areas. A two-level hierarchical game is formulated in which the choices made by the PEV customers in choosing a particular charging station is modeled as a lower-level congestion game and the price competition among stations is modeled as the upper-level game. The Nash equilibrium is obtained. The basic tool to study this problem is game theory. Each station is modeled as a player who has to decide which customers to attract based on its location by adjusting its price, in reaction to the other player’s strategy. We identify conditions under which a Nash equilibrium exists and characterize the strategies at equilibrium selection of locations. The results are summarized in chapter 4.

In chapter 5, we consider the problem of pricing and scheduling of electric vehicle charging trajectories by a charging station owner when some electric vehicles offer the added flexibility of temporary discharge. The solution of the problem involves real-
time planning of the PEV charging profile and the market equilibrium in the dynamic setting. Unlike most traditional literature that is on static decisions associated with finite invariant prices, we determine our prices as function of time and charging trajectories. Our results characterize the trajectory at a competitive equilibrium that maximizes both the customer and the station’s utilities. Besides the competitive equilibrium as a result of the competition between price-taking players, the scenario when the station has sufficient market power to determine the price is also addressed.

1.3 Literature Review

In this section, we provide a high-level overview of the literature of the areas related to the work in this dissertation.

The distributed systems were initially studied around 1970s [12] under the terminology ‘decentralized system’. The limit of LTI state feedback control of the decentralized systems are studied in [1, 3, 77], where a significant concept ‘fixed mode’ is proposed. This problem is further addressed in time-varying or periodic control laws in [2, 39]. In [31, 32], the information transmission nature, defined as ‘signaling’ is discovered, which pointed out that the stabilization of the decentralized systems requires transmission of information through plant. Such a signaling process is initiated by injection of the control input by one control station and then received or decoded by observations of another control station. The aggregation of paths of a series of signaling processes in a large scale system is also modeled as a communication network [61]. Optimal design of the signaling processes are addressed in [81].

The topic of networked control has also been widely studied, an overview of this area can be found in [3, 26]. The impact of analog erasure channel on the stability and performance of the control systems are studied in [21, 70]. The data rate constraint of the lossless channels in the networked control systems is considered in [49, 54, 55]. As the wireless communication becomes more and more popular, there are also a
lot of works that considers the control systems with feedback information sent over additive white Gaussian noise (AWGN) channels. Discussion on Gaussian point-to-point channel can be found in [7, 14]. Results are also extended to Gaussian MAC channel [83], Gaussian product channel [35], a Gaussian relay channel [34].

An overview of electric vehicle charging technology and its impact and integration into the existing power systems is provided in [46, 65, 71]. Locations of the PEV charging stations are considered in [25]. More general study of facility locations can be found in transportation or operations research literature [28, 36, 62, 63, 79] The locations of power facility location are discussed in [6].

The main theoretic tool in the market study of this dissertation is game theory. The book by Fergusson [15] offers a concise description of various games in the market competition such as Cournot, Bertrand and Stackelberg games. This dissertation also relies heavily on the congestion game model. The work in [51, 52] provides a thorough research in the nonatomic congestion game. The queueing effect in the pricing game can be found in [9, 11]. Recent advances in the potential games can be found in [47].

The concept of competitive equilibrium can be found in economic text such as [4]. The application of competitive equilibrium in power system ancillary service pricing can be found in [75]. The dynamic competitive equilibrium and its application in power market is in [76]. The planning of charging trajectories in the electric vehicle batteries is studied in [33, 78]. Dynamic pricing algorithms based on inventory are discussed in [19].
CHAPTER 2

STABILIZING DECENTRALIZED SYSTEMS ACROSS ANALOG ERASURE LINKS

2.1 Introduction

Designing decentralized control laws to stabilize a system has a long history. It is known that a linear time invariant (LTI) output feedback control law cannot stabilize a system with unstable fixed modes \cite{2,12,77} and to stabilize such a system, either linearity or time-invariance should be sacrificed. A good summary of related results and extensions can be found in \cite{20,31} and the references therein. In \cite{31}, a necessary and sufficient condition was obtained for controllability of a system using a decentralized control structure. It was shown that through an appropriate signaling strategy, controllers can shrink their local unobservable and / or uncontrollable subspaces, and thus, jointly control the system. In this context, signaling implies treating the process that is being controlled as a communication channel and transmitting information across it from one controller to another.

However, most of this classical work assumes that any information transmission from the sensors to the controllers and from the controllers to the actuators occurs over perfect external channels. In this chapter, we consider decentralized stabilization of a system in which the components communicate across links that may stochastically drop any data transmitted across them. Presence of stochastic packet drops complicates the problem substantially since we do not assume existence of acknowledgements for successful transmission along any channel. As a result, the transmitter
can never be sure that the signaled information has been successfully received. The
transmitter and the receiver may, thus, become ‘desynchronized’ in terms of how
much information has been transmitted. In particular, since the signaling algorithms
required for decentralized stabilization, such as the algorithms proposed in [20, 31],
rely on a sequence of information transmissions, they may not be directly applicable
to this problem. The chief contribution of this chapter is the identification of con-
ditions on the process and the external communication channels that are necessary
for the process to be stabilized in a decentralized manner. Conversely, we provide
an algorithm that allows decentralized stabilization of the process if these conditions
are satisfied.

Most of the work on control across communication channels that erase packets
stochastically (also known as analog erasure links) has focused on centralized con-
trol – namely the case when there is only one controller present, which obviates the
need for signaling in the sense described above. In this setting, if the analog erasure
links are present between the sensor and the controller, desynchronization between
the transmitter and the receiver is not important since the optimal strategy for the
sensor is to transmit the latest data irrespective of whether or not the old data were
received successfully [23, 24, 45, 70]. However, when the analog erasure link is present
between the controller and the actuator, the data that needs to be transmitted by the
controller may depend on whether the previous transmission was successful or not.
Thus, even in this centralized setting, desynchronization between the transmitter and
the receiver makes stabilization more difficult [70]. It is only recently that the stabi-
lizability conditions have been identified without the presence of acknowledgements
for this channel [22]. We provide such stabilizability conditions for decentralized
systems, where signaling provides an additional complication.

There have not been many works that have considered distributed control and
imperfect communication channels simultaneously. There are some works when the
communication channel model adopted is that of a digital noiseless channel \[55, 80, 85\]. However, such a channel model does not display the desynchronization effect mentioned above and hence is not directly relevant to our problem. The work in \[38\] considers control design for distributed systems in the presence of analog erasure links, but presents only sufficient conditions to achieve a given level of performance and does not consider stabilizability. The recent work \[66\] presents an interpretation of the signaling required for decentralized stabilization in terms of information from an unstable mode to itself through the process. Our work uses a similar interpretation of signaling through the process; however, the presence of analog erasure links presents an additional complication.

The main result of this chapter is a necessary condition for stabilizability of a decentralized control system in which analog erasure channels connect sensors to controllers, and the controllers to actuators. By constructing a specific algorithm, we also prove that this condition is sufficient for stabilizability. We solve the desynchronization problem mentioned above by constructing an algorithm to re-synchronize the system periodically. For the extreme case of only one sensor, controller, and actuator, the problem reduces to that of centralized control and our results recover the conditions in \[21, 22, 70\]. Alternatively, for the case when the erasure probability in every channel is zero (i.e., all the channels are reliable), we recover the classical framework of \[31\]. In this sense, our results form a bridge between the two streams of work.

The chapter is organized as follows: Section 2.2 formulates the problem. Section 2.3 states the main result of this chapter, which is proved in Section 4.4. For pedagogical ease, the proof is first presented for the case when no measurement and process noises exist (Section 2.4.2) and then extended to the general case (Section 2.4.3). Section 5.6 concludes the chapter. The results of this chapter can be seen in \[41\].
**Notation:** The space of \(n\)-dimensional real vectors is denoted by \(\mathbb{R}^n\). The eigenvalues of an \(n \times n\) matrix \(A\) are denoted by \(\{\lambda_i(A)\}_{i=1}^n\), and its spectral radius by \(\rho(A)\). A lowercase letter with a subscript, such as \(x_i\), denotes the \(i\)-th element of the vector \(x\). A lowercase letter with a superscript, e.g. \(x^{(i)}\), denotes the \(i\)-th vector in a sequence of ordered vectors.

### 2.2 Problem Statement

**Process and Control Stations:** Consider a discrete time linear time-invariant process with \(M\) control inputs (each applied by a different control station) that evolves as

\[
x(k + 1) = Ax(k) + \sum_{i=1}^{M} B_i u^{(i)}(k) + w(k),
\]

where \(x(k) \in \mathbb{R}^n\) is the state of the process, \(u^{(i)}(k) \in \mathbb{R}^{m_i}\) for \(i = 1, 2, \ldots, M\) is the control input applied by the \(i\)-th control station and \(w(k) \in \mathbb{R}^n\) is the process noise. Each control station \(i\) (\(1 \leq i \leq m\)) consists of a sensor \(S_i\), a controller \(C_i\), and an actuator \(A_i\). At each time \(k\), the sensor \(S_i\) generates a measurement \(y^{(i)} \in \mathbb{R}^{t_i}\) according to the equation

\[
y^{(i)}(k) = C_i x(k) + v^{(i)}(k).
\]

Define the overall sensing matrix \(C\) and the overall actuation matrix \(B\) for the system as

\[
C \triangleq \begin{bmatrix} C_1^T & C_2^T & \cdots & C_m^T \end{bmatrix}^T, \quad B \triangleq \begin{bmatrix} B_1 & B_2 & \cdots & B_m \end{bmatrix}.
\]

Given matrices \(A\), \(B_i\) and \(C_i\), define \(\mathcal{R}_i\) as the controllable subspace and \(\mathcal{K}_i\) as the unobservable subspace for station \(i\).
Assumption 1 (Noises and Initial Vector).

(1) All the noises \( w(k) \in \mathbb{R}^n \) and \( v^{(i)}(k) \in \mathbb{R}^{l_i} \) \((1 \leq i \leq M)\) are assumed to be random vectors with elements distributed according to an arbitrary probability density function with bounded support and zero mean.

(2) The covariance matrices of the noise vectors are diagonal so that for \( i \neq j \), \( \mathbb{E}w_i(k)w_j(k) = 0 \) and \( \mathbb{E}v_i^{(l)}(k)v_j^{(l)}(k) = 0 \) for \( l = 1, \ldots, M \) and any \( k \).

(3) The initial condition \( x(0) \) is a random vector with zero mean and \( \mathbb{E}x_i(0)x_j(0) = 0 \) for \( i \neq j \).

Assumption 2 (System Structure).

(1) The matrix \( A \) is in the Jordan form with real eigenvalues. The case with complex eigenvalues is similar but notationally more involved.

(2) Similar to the assumption made in [49, 55], the geometric multiplicity of every eigenvalue is equal to 1.

(3) All eigenvalues of \( A \) lie outside the unit circle, so that no mode of the system is stable without input from at least one control station.

(4) The pair \( (A, C) \) is observable and the pair \( (A, B) \) is controllable.

Channel Model and Topology: We use the analog erasure channel model for any explicit communication channel in the problem. An analog erasure channel ignores quantization effects and concentrates on packet drops. Specifically, the channel supports an input \( i(k) \in \mathbb{R}^p \) of a bounded (and constant) dimension \( p \) at every time \( k \). The output \( o(k) \) of the channel at time step \( k \) is given by

\[
o(k) = \begin{cases} 
  i(k) & \text{with probability } 1 - \varepsilon \\
  \varphi & \text{otherwise,}
\end{cases}
\]

(2.3)

where \( \varphi \) denotes that the receiver knows that an erasure event has occurred at the current time. The probability \( \varepsilon \) is called the erasure probability of the channel. For simplicity, we assume that the transmission across the channel can be completed within a sampling period [23, 70].
At every time $k$, in every control station $i$, the sensor $S_i$ transmits data to the controller $C_i$ across a dedicated analog erasure channel. Similarly, the controller $C_i$ transmits data to actuator $A_i$ across a separate analog erasure channel. Note that there is no explicit communication among the control stations. Denote the set of erasure probabilities for the $2M$ analog erasure channels as $\mathcal{E}$.

**Assumption 3** (Mutual Independence). *The erasure event processes on the various analog erasure channels are mutually independent. Further, all the primitive random variables in the system ($w(k)$, $v^{(i)}(k)$'s, $x(0)$ and the erasure events) are mutually independent, which means, in particular, that the process and measurement noises are white.*

*Remark 2.2.1.* It is worth pointing out that because of the presence of analog erasure channels, the input $u^{(i)}(k)$ in (2.1) should be interpreted as the control input applied by $A_i$, rather than that generated by $C_i$.

**Assumption 4** (Computational Capability). *To concentrate on the effect on stabilization in the presence of the erasure channels, we assume that every controller and sensor has access to the system model and has unlimited computational power. However, to prevent the actuators from assuming the role of the controllers, we assume that the actuators do not have access to the system model and are only able to perform logical operations, such as buffer retrieval (see also [22]).*

Since multiple actuators can potentially control a single mode, we need a coordinating mechanism to arbitrate between them.

**Assumption 5** (Actuator Coordinating Mechanism). *An actuator coordinating mechanism exists that buffers all incoming signals sent to the actuators and releases them to the intended actuators according to a static priority scheme. Note that implementing this mechanism does not require access to the plant model and is in keeping with Assumption 4.*
Stabilizability: We say that the system \((2.1)\) is stable in the mean square sense if and only if

\[
\limsup_{k \to \infty} E[x^T(k)x(k)] < c,
\]

where \(c\) is a positive constant. If there exist design parameters such that the closed loop system is stable in this sense, the system is stabilizable. The design parameters are the information that each sensor \(S_i\) transmits to the controller \(C_i\), the information that \(C_i\) transmits to the actuator \(A_i\), and the logical operation that each actuator performs. Note that these parameters implicitly include the controller design.

Problem Statement: We wish to obtain conditions on the set of erasure probabilities \(\mathcal{E}\) and the system matrices under which the system \((2.1)\) is stabilizable.

2.3 Main Result

Our main result is based on the characterization of the implicit data flow in the system \((2.1)\). Define the \(n\) eigenvalues of \(A\) by \(\{\lambda_i| i = 1, \ldots, n\}\) and the corresponding eigenvectors by \(\{\Lambda_i, i = 1, \ldots, n\}\). The state \(x(k)\) at any time \(k\) can be decomposed as \(x(k) = \sum_{i=1}^{n} \alpha_i(k)\Lambda_i\). The component \(\alpha_i(k)\Lambda_i\) is the value of the \(i\)-th mode of the system at time \(k\). Denote the modes of the system by \(\{x(i)| i = 1, \ldots, n\}\) and by abusing the notation somewhat, the value of the \(i\)-th mode at time \(k\) by \(x^i(k)\).

The mode \(x^{(m)}\) is unstable if the corresponding eigenvalue \(\lambda_m\) lies outside the unit circle. For every mode \(x^{(m)}\) of the system \((2.1)\), define

- an actuator set \(D_m = \{A_i| i : x^{(m)} \in \mathcal{R}_i\}\) to represent actuators that can ‘drive’ \(x^{(m)}\).
- a sensor set \(Q_m = \{S_i| i : x^{(m)} \in \mathcal{K}_i^\perp\}\) as the set of sensors that can observe \(x^{(m)}\).

If there is no \(i\) such that \(x^{(m)} \in \mathcal{R}_i \cap \mathcal{K}_i^\perp\), the mode \(x^{(m)}\) cannot be stabilized by any one control station acting alone. In this case, the information about the
value of this mode needs to be signaled through the plant from the sensors in $Q_m$ to the control stations connected to the actuators in $D_m$, through the communication network defined together by the system and the external analog erasure channels. We say that control station $i$ is connected to control station $j$, and denote it as $j \rightarrow i$, if $R_j \not\subset K_i$. Intuitively speaking, this condition implies that at least one mode controllable from station $j$ is observable from station $i$. Thus, this mode can be used to transmit information from the control station $j$ to control station $i$ through signaling [2]. Define a path of control stations $\mathcal{P}_{j_1,j_p}$ from station $j_1$ to station $j_p$ as the set (if it exists) of stations $j_1, j_2, \ldots, j_p$ such that $j_1 \rightarrow j_2 \rightarrow \ldots \rightarrow j_p$ and all $j_i$’s, $i = 1, \ldots, p$ are distinct. Since the control systems are cooperating with one another, if there is a path from $j_1$ to $j_p$, then $j_1$ can transmit information to $j_p$ through repeated signaling. Finally, if there exists a path $\mathcal{P}_{i,j}$ for any two stations $i$ and $j$ (with $i \neq j$), we say that the system in (2.1) is strongly connected.

**Assumption 6.** System (2.1) is strongly connected.

Assumption 6 was shown in [12, 20] to be sufficient for the system (2.1) to be stabilizable if the explicit communication channels are perfect. We assume that this condition holds to study the additional burden on stabilizability imposed by the erasure channels.

Assumption 6 ensures the existence of a data flow network for every unstable mode, constructed as follows. For every mode $x^{(m)}$, define a directed graph $\mathcal{G}_m = (\mathcal{V}_m, \mathcal{E}_m)$. Every node of the graph represents a distinct device (sensors, controllers or actuators) in the system. We will use the terms node and device interchangeably.

The edge set is constructed as follows. For every pair of a sensor $S_i \in Q_m$ and an actuator $A_j \in D_m$, consider all possible paths $\mathcal{P}_{ij}$ with the constraint that there is no intermediate node $k$ for which either condition $S_k \in Q_m$ or $A_k \in D_m$ holds. For each such path $\mathcal{P}_{ij} = \{i, j_1, j_2, \ldots, j\}$ with $i \rightarrow j_1 \rightarrow j_2 \ldots \rightarrow j$, introduce directed edges from the node corresponding to sensor $S_i$ to the one corresponding
to controller $C_i$, from $C_i$ to $A_i$, from $A_i$ to $S_j$, and so on till the edge from $C_j$ to $A_j$. By considering all the possible paths from $i$ to $j$ and all node pairs $(i, j)$, we obtain the edge set $E_m$. The edges that connect the sensors to the controllers and the controllers to the actuators correspond to external communication channels and are assigned the erasure probability of the corresponding channel from the set $E$. The edges that connect the actuators to the sensors are signaling channels and are assigned an erasure probability of 0.

**Example 1.** Consider the process

\[ x_1(k+1) = \lambda_1 x_1(k) + b_1 u_1(k) \]
\[ x_2(k+1) = \lambda_2 x_2(k) + b_2 u_2(k) \]
\[ x_3(k+1) = \lambda_3 x_3(k) + b_3 u_3(k) \]

begin observed by three sensors

\[ y_1(k) = [0 \ 1 \ 1] x(k) + v^{(1)}(k), \]
\[ y_2(k) = [1 \ 0 \ 0] x(k) + v^{(2)}(k), \]  \[ y_3(k) = [1 \ 0 \ 0] x(k) + v^{(3)}(k). \]

where $x(k) = [x_1(k), x_2(k), x_3(k)]^T$. Figure 2.1 illustrates the graph $G_1$ for mode $x^{(1)}$. In the figure, the solid arrows represent the analog erasure links and the dashed arrows represent the signaling channels. We also include two nodes labeled as the mode $x^{(1)}$ in the graph to indicate the source and destination of the information flow.

For the communication graph $G_m$, define a cut-set as a division of all the nodes in the network into two sets: (i) a source set that contains, in particular, all the nodes corresponding to the sensors $S_i \in Q_m$, and (ii) a sink set that contains, in particular, all the nodes corresponding to the actuators $A_j \in D_m$. For each cut-set, define the
cut-set probability as the product of the erasure probabilities for all the edges from any node in the source set to any node in the sink set. Finally, define the max-cut erasure probability of the graph $G_m$ as the maximum cut-set probability among all the possible cut-sets for the graph. The following is the main result of this chapter.

**Theorem 2.3.1.** Consider the problem formulation in Section 2.2. For each unstable mode $x^{(m)}$ of the system, construct the graph $G_m$ and let $p_{\text{max-cut}(G_m)}$ be the max cut erasure probability for this graph. The decentralized control system (2.1) can be stabilized if and only if for every unstable mode $x^{(m)}$

$$p_{\text{max-cut}(G_m)}|\lambda_m|^2 < 1. \quad (2.7)$$

2.4 Proof of the Main Result

We begin by proving the necessity of the condition (2.7). The sufficiency of the condition is then proved in two steps. In section 2.4.2, we prove the sufficient condition in the case if there are no process and measurement noises in the system. Then, we proceed to the case when bounded noises are present in section 2.4.3.
2.4.1 Proof of Necessity

*Proof of necessity for Theorem 2.3.1.* The system (2.1) being stabilized is equivalent to all modes $\{x^{(i)}\}, i = 1, \ldots, n$ being stabilized. By construction, $G_m$ is the union of all possible paths that can transmit information about the value of the mode $x^{(m)}$ to the actuators that can control the mode. The evolution of this mode is governed by the Jordan block $J_m$ of the matrix $A$ that corresponds to the eigenvector $\Lambda_m$.

Construct a graph $G_m^*$ by starting with $G_m$ and performing the following actions:

1) Identify a cut-set in $G_m$ for which the cut-set probability is equal to $p_{\text{max cut}}(G_m)$.

2) For this cut-set, introduce a perfect link (i.e., link with erasure probability 0 and no delay) between every pair of nodes in the source set of $G_m$. Since, in particular, all the sensors that can observe $x^m$ in the set $Q_m$ are in the source set, this operation implies that all nodes in the source set are provided access to the data generated by all the sensors at that time step. Thus, effectively, there is only one sensor present.

3) Similarly, introduce a perfect link between every pair of nodes in the sink set. Since, in particular, all the actuators in the set $D_m$ are in the sink set, this operation implies that there is effectively only one actuator in the sense that all actuators have the same information about the mode value $x^m(k)$. Further, allow the nodes in the sink set thus obtained to have knowledge of the state model and unlimited computational power. Since the communication graph $G_m$ is a subgraph of $G_m^*$ and in $G_m^*$, the components have more knowledge and computation power than in $G_m$, a necessary condition for the mode $x^{(m)}$ to be stabilized across $G_m$ is that it can be stabilized across $G_m^*$. Now, by relabeling the source set as the sensor and the sink set as the controller, we realize that stabilization of the mode $x^{(m)}$ across the graph $G_m^*$ is a special case of the problem studied in [21]. Specifically, we are interested in a necessary condition for stabilization for a linear time-invariant system in which the sensor is connected to the controller through a network of erasure channels with its max cut erasure probability $p_{\text{max cut}}(G_m)$. For this situation, Proposition 5 in [21] states that a
necessary condition for any algorithm to stabilize this mode is \(p_{\text{max, cut}}(\lambda_m)|\lambda_m|^2 < 1\).

By considering all modes of the system, we prove the necessity of Theorem 2.3.1.

2.4.2 Proof of Sufficiency in the Absence of Noise

We now prove the sufficiency of the condition in Theorem 2.3.1 for stabilization of the system (2.1) when no process and measurement noises are present. We construct a specific algorithm for this purpose. The algorithm relies on systematic signaling of information through the plant from one control station to another. The signaling strategy we use is based on the following structural result.

Lemma 2.4.1. (From [81, Lemma 3.1]) Consider control stations \(i\) and \(j\). The condition \(i \rightarrow j\) holds if and only if there exist at least one row \(c_{j\theta}\) of \(C_j\), one column \(b_{i\eta}\) of \(B_i\) and an integer \(l_{ij}\) with \(1 \leq l_{ij} \leq n\) such that \(c_{j\theta}A^{l_{ij}}b_{i\eta} \neq 0\).

For the next few results, we assume no packet erasure to describe the signaling algorithm.

Lemma 2.4.2. Consider any integer \(\beta\) such that \(1 \leq \beta \leq n\). There exist a sequence of transmissions from the various devices such that at any time \(k > n\), any control station \(j\) can compute the value \(c_{j\beta}A^kx(0)\) for any integer \(k > n\), where \(c_{j\beta}\) is the \(\beta\)-th row of \(C_j\).

Proof. Let the system (2.1) evolve in open loop for time \(i = 0, \ldots, n - 1\). Station \(j\) records the first \(n\) outputs \(y_{j\beta}^{(0)}, y_{j\beta}^{(1)}, \ldots, y_{j\beta}^{(n - 1)}\). According to Caley-Hamilton theorem, \(A^n = \sum_{i=0}^{n-1} \alpha_i A^i\), for some parameter \(\alpha_i\)'s. Given \(k > n\), we
have
\[ c_{j\beta}A^kx(0) = c_{j\beta} \left( A^{k-n}A^n \right) x(0) = c_{j\beta} \left( A^{k-n} \sum_{i=0}^{n-1} \alpha_i A^i \right) x(0) \]
\[ = c_{j\beta} \sum_{i=0}^{n-1} \alpha'_i(k)A^i x(0) = \sum_{i=0}^{n-1} \alpha'_i(k)y_{\beta i}^{(j)}(i), \]
for an appropriately calculated set of parameters \( \{\alpha'_i(k)\} \). Therefore, the term \( c_{j\beta}A^kx(0) \) can be computed by the control station given the values \( y_{\beta i}^{(j)}(0), \ldots, y_{\beta i}^{(j)}(n-1) \).

Denote by \( n_{i\eta} \) the dimension of the controllable subspace of the pair \( (A, b_{i\eta}) \).

**Lemma 2.4.3.** Let \( i \rightarrow j \). Then, there exist a sequence of transmissions starting from any time \( \bar{k} > n - l_{ij} - 1 \) such that the \( C_i \) can transmit any desired scalar \( \gamma \) to \( C_j \).

**Proof.** Since \( i \rightarrow j \), Lemma 2.4.1 implies that there exist parameters \( \eta, \theta \) and \( l_{ij} \) such that \( c_{j\theta}A^{l_{ij}}b_{i\eta} \neq 0 \). The input \( u^{(i)}(\bar{k}) \) is set to be a zero vector except the \( \eta \)-th component that is set to be \( u^{(i)}_{\eta}(\bar{k}) = \gamma/(c_{j\theta}A^{l_{ij}}b_{i\eta}) \). For all other times \( k \), \( 0 \leq k \leq \bar{k} + 1 + l_{ij} \), set \( u^{(j)}(k) = 0 \). Further, set all control inputs from every other station except \( i \) to be 0 for all time steps. At time \( \bar{k} + 1 + l_{ij} \), the control station \( j \) observes
\[ y_{\theta j}(\bar{k} + 1 + l_{ij}) = c_{j\theta} \left( A^{k+1+l_{ij}}x(0) + A^{l_{ij}}b_{i\eta}u_{i\eta}(\bar{k}) \right) \]
\[ = c_{j\theta}A^{k+1+l_{ij}}x(0) + \gamma. \]  
(2.8)

Since station \( j \) can compute \( c_{j\theta}A^{k+1+l_{ij}}x(0) \) according to Lemma 2.4.2, it can compute \( \gamma \) at time \( \bar{k} + 1 + l_{ij} \).

**Remark 2.4.4.** Note that signaling the value \( \gamma \) at time \( k \) alters the state \( x(k) \) for future times. To eliminate the effect of the signaling, control station \( i \) can construct
an input sequence \( u_i^{(l)}(k + 1), \ldots, u_i^{(l)}(k + 1 + n_{i\eta}) \) such that the state of process at time \( n_{i\eta} + 1 \) is \( x(n_{i\eta} + 1) = A^{n_{i\eta}+1}x(k) \), which is independent of \( \gamma \). From now on, we assume that once the control station \( i \) has signaled any value to another station \( j \), it cancels the effect of this signaling. Notice that if \( n_{i\eta} < l_{ij} \), station \( i \) must wait for at least \( l_{ij} + 1 \) steps before canceling the signaling effect to make sure that station \( j \) correctly decodes \( \gamma \).

Remark 2.4.5. The value \( \gamma \) may be the initial condition of a mode \( x_m(0) \). In our stabilization algorithm, we will assume that if the initial condition of a mode is sought to be signaled, then a unique identifier corresponding to the mode \( m \) is transmitted first. This convention resolves any ambiguity in the information received by a control station that seeks to decode the transmissions from control station \( i \). The effect of the two transmissions can be canceled as described above.

Remark 2.4.6. It is possible that actuators corresponding to multiple control stations may transmit information to the same control station simultaneously. For instance, in Figure 2.1, \( A_1 \) and \( A_2 \) may apply signaling inputs at the same time. In this case, \( S_1 \) will be unable to decode the information from either actuator. To prevent this event, all the incoming signaling sequences are buffered and released by the coordinating mechanism to the intended actuators serially. The priority for various actuators is decided a priori and known to all the control stations.

Repeated application of Lemma 2.4.3 yields the following result.

Lemma 2.4.7. Let \( i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_{n'} \) with \( S_i \in Q_m \) and \( A_{i_{n'}} \in D_m \) for some mode \( m \). Then, there exists a sequence of transmissions such that \( S_i \) can transmit any value \( \gamma \) to \( A_{i_{n'}} \).

Remark 2.4.8. Since the system matrix \( A \) is in the Jordan form, if the mode value \( x^{(m)}(k) \) is known to the control station \( C_j \), it can construct a sequence of control inputs that can drive this mode to 0 without affecting the evolution of the other
modes. We call this input sequence as the stabilizing sequence for the mode \( m \).

**Remark 2.4.9.** There may exist a mode \( x^{(m)} \in \mathcal{R}_i \cap \mathcal{R}_j \) for two stations \( i \) and \( j \), \( i \neq j \). If actuators \( A_i \) and \( A_j \) apply the stabilizing sequence for the mode \( m \) whether simultaneously or sequentially, the mode \( x^{(m)} \) will not be driven to the origin. To solve this problem, the indices of the modes whose stabilizing sequences have been received from the various controllers are recorded at the actuator coordinating mechanism. If a stabilizing sequence for the same mode arrives later, it is discarded.

The above discussion pertain to the case when there are no erasures. When there are erasures, we will use the same algorithms for signaling and stabilization as described above; however, every transmission will be repeated \( M \) times, where \( M \) is a design parameter. Clearly, as \( M \) increases, the probability of a successful transmission also increases.

**Stabilizing Algorithm:** The algorithm operates in a batch-to-batch manner. Each batch has a length of \( L \) steps, where \( L \) is a parameter chosen by the designer. A batch is divided into three consecutive and non-overlapping phases:

1. **Observation:** No control inputs are applied during this phase of length \( n_0 = \max\{\dim \mathcal{K}_i^j\} \). Every sensor \( i \) uses the observations it receives to deduce the initial condition \( x_m(0) \) for every mode \( m \) for which \( S_i \in Q_m \).

2. **Data transmission:** This phase of length \( N_s \) (which is a design parameter) is used for the transmission of the information about every mode \( x_m(0) \) from the sensors in \( Q_m \) to the actuators in \( D_m \). During this phase, for every mode \( m \) and over every path \( i_1 \rightarrow i_2 \rightarrow \ldots \rightarrow i_n \), for this mode with \( S_{i_1} \in Q_m \), and \( A_{i_n} \in D_m \), the mode initial condition \( x_m(0) \) is transmitted by the sensor \( S_{i_1} \) to the controller \( C_{i_n} \) and the stabilizing sequence for this mode is transmitted from \( C_{i_n} \) to \( A_{i_n} \) at every time step. For external channels, the messages are transmitted simultaneously using the

\[ ^1 \text{This stabilizing sequence will be applied during stage III and hence is calculated by predicting the mode value as it evolves in open loop till the time step when the sequence is scheduled to be} \]
fact that the channel supports a vector at every time step. For signaling channels, the actuator coordinating mechanism imposes a schedule so that two actuators do not change the plant state simultaneously. Every stabilizing sequence that is successfully received is stored by the actuator coordinating mechanism. As mentioned above, whenever the transmission is across an erasure channel, it is repeated $N_s$ times.

(3) Stabilization: In this phase, the stabilizing sequences held in phase II are released one by one by the actuator coordinating mechanism. Denote the length of this phase by $n_a$. Actuators apply the released stabilizing sequences to stabilize the system. Every mode is stabilized in sequence, with $n$ steps allocated to it. If there is no stabilizing sequence corresponding to a node buffered by the actuator coordinating mechanism, that node is allowed to evolve in open loop.

Thus, every batch of the stabilization algorithm is of length $L = n_a + N + n_a$. The stabilization algorithm consists of repeating these batches. In the $k$-th batch, during the observation phase, the sensors observe the initial conditions $x^m((k-1)L)$ and the control stations seek to stabilize them.

Proof. A sufficient condition for the system to be stable is that every mode is stabilized. Since there is no process or measurement noise, if the stabilizing sequence corresponding to a particular mode is received by the actuator coordinating mechanism, by the end of phase II in any batch, the mode value is set to the value of zero by the end of phase III and the mode is stable from that time onwards. Note that the algorithm we propose transmits information from sensors in $S_m$ to the actuator coordinating mechanism across the graph $G_m$. The communication channels from actuator nodes to sensor nodes in $G_m$ have erasure probability zero; however, they introduce a finite and constant delay. We can construct another graph $G_m'$ that is identical to $G_m$, except for the introduction of ‘dummy’ edges in place of these applied.
signaling channels, such that all these edges have erasure probability 0, and incur a delay of one time step. Given the stabilization protocol above and the sequence of erasure events, the actuator coordinating mechanism receives the same data across the graph $G'_m$ as the graph $G_m$, and hence the stabilization conditions are equivalent across the graph $G'_m$. But we note that the algorithm for transmitting information about mode $m$ across $G'_m$ is the same as the algorithm proposed in [21] for control of a plant across a network with erasure channels. Since the mode $m$ evolves according to the system matrix $J_m$, [21, Proposition 6] yields that the mode $m$ will be stable if $p_{\text{max, cut}}(G_m)|\lambda_{\text{max}}(J_m)|^2 < 1$. Since (2.7) ensures that this condition is met for every mode $m$, it is a sufficient condition for stability.

\[ \square \]

2.4.3 Proof of Sufficiency with Bounded Noise

We now derive the sufficient condition of the system (2.1) with process and measurement noise being present. As stated in section 2.4.2, we assume that these noises have bounded support; in particular, $\|w(k)\| \leq M_1$ and $\|v_j(k)\| \leq M_{2,j}$. The stabilizing algorithm is the same as before except that the stations transmit the state value at time $(K - 1)L$ during the $K$-th batch, instead of transmitting the initial state at every step. Further, for signaling when $i \rightarrow j$, we use the fact that since the process and measurement noises are bounded, the output at the station $j$, denoted as $y_{j\theta}$, is also bounded when the station $i$ does not apply any input. If the station $i$ chooses an input large enough to make $y_{j\theta}$ exceed the bound, station $j$ can always detect a transmission from station $i$.

To transmit a value $\gamma$ to station $j$ when the bound of the output $y_{j\theta}$ in the absence of input from station $i$ is $Y_j$, station $i$ quantizes $\gamma$ using an encoding function...
$f_{\text{enc}} : \mathbb{R} \rightarrow \mathbb{Z}$ for $l \in \mathbb{Z}^+$ and quantization level $\Delta > 0$,

$$f_{\text{enc}}(\gamma) = \begin{cases} 
  l, & (l-1)\Delta \leq \gamma < l\Delta, \\
  -l, & -(l-1)\Delta \leq \gamma < -(l-1)\Delta.
\end{cases} \quad (2.9)$$

The signaling input $u_i(k)$ is selected as

$$u_{it}(k) = \begin{cases} 
  2^{f_{\text{enc}}(\gamma)}Y_j/(c_jA_l^ib_{l\eta}) & \text{if } t = \eta \\
  0 & \text{otherwise},
\end{cases} \quad (2.10)$$

where $u_{it}(k)$ represent the $t$-th element of $u_i(k)$ and $l_{ij}$ was defined in Lemma 2.4.1.

The sensor estimates the output $y_{j\theta}(k)$ using the open loop system model and computes the estimation error $\tilde{y}_{j\theta}(k)$ at every $k$. If the input $u_{it}(k)$ is applied, the magnitude of the error $|\tilde{y}_{j\theta}(k)|$ will lie in the interval $[(2l-1)Y_j, (2l+1)Y_j]$. Station $j$ can thus decode $l$ using the decoding function $f_{\text{dec}} : \mathbb{Z} \rightarrow \mathbb{R}$:

$$f_{\text{dec}}(\tilde{y}_{j\theta}(k)) = \begin{cases} 
  l & (2l-1)Y_j \leq \tilde{y}_{j\theta}(k) \leq (2l+1)Y_j \\
  -l & -(2l+1)Y_j \leq \tilde{y}_{j\theta}(k) \leq -(2l-1)Y_j.
\end{cases} \quad (2.11)$$

By the encoding and decoding strategies (2.9)(2.11), station $j$ can recover $\gamma$ with a quantization error bounded by $\Delta/2$.

We now show that with this change in the signaling strategy, the algorithm proposed in the noiseless case will also stabilize the process with noise.

Proof. We need only to show that the mode $m$ can be driven to a bounded value with the received quantized feedback information in the noisy case. Denote the event that the the control sequence is applied at $K$-th batch by $\mathcal{E}_1$. The covariance of mode $x_m$ under this event can be bounded by

$$\mathbb{E}\{x_m^2(KL)|\mathcal{E}_1\} \leq |\lambda_{\text{max}}(J_m)|^{2L}|c_m((K-1)L)|^2 \leq \lambda_m^2M_m^2,$$

where $M_m > 0$ is the bound of the quantization error. The covariance of mode $x_m$,
denoted by $P_m(k)$ thus evolves as

$$
P_m((K + 1)L) \leq p_F |\lambda_{\text{max}}(J_m)|^{2L}P_m(KL) + (1 - p_F)\lambda_m^{2L}M_m^2. \quad (2.12)
$$

where $p_F = \Pr\{\mathcal{E}_1^c\}$ with $\mathcal{E}_1^c$ being the complement of $\mathcal{E}_1$. An argument similar to the one outlined in [21, Proposition 6], then yields that

$$
\lim_{L \to \infty} p_F^{\frac{1}{2}} |\lambda_{\text{max}}(J_m)|^2 = p_{\text{max, cut}}(G_m)|\lambda_{\text{max}}(J_m)|^2. \quad (2.13)
$$

Since the second term in (2.12) is a bounded quantity uncorrelated with $P_m(k)$, we can ensure that $P_m(k)$ is bounded if $p_{\text{max, cut}}(G_m)|\lambda_{\text{max}}(J_m)|^2 < 1$ by choosing $L$ large enough. For $P_m(k)$ with $k \in [(K - 1)L, KL]$, since $x_m(k)$ grows in open loop for only $L - 1$ steps in every batch, once the value $P_m(KL)$ is bounded, the whole sequence $P_m(k)$ is also bounded. Since (2.7) ensures that this condition is met for every mode $m$, it is a sufficient condition for stability.

2.5 Conclusions

In this chapter, decentralized control system over both erasure and signaling channels is studied. Explicit communication graph is proposed to study the feedback information flow for stabilization. A necessary and sufficient condition for its stabilizability is obtained. Stabilizing algorithms are constructed to prove the sufficient condition.
CHAPTER 3

ON STABILIZABILITY OF LTI SYSTEMS ACROSS A GAUSSIAN MAC CHANNEL

3.1 Introduction

Networked control systems have attracted much attention recently \cite{5,26}. There is a lot of literature on systems whose components (plant, controller, sensor and actuator) are connected using point-to-point channels. Stabilizability and performance of such systems are influenced by the constraints introduced by the communication channel such as packet drops \cite{23,24}, channel delay \cite{59,69}, limited data rate \cite{53,56,74}, and so on. Particularly relevant to this chapter is the stream of work that has considered stabilization when the sensor-controller or controller-actuator communication occur across an additional white Gaussian noise (AWGN) channel. A tight data rate bound has been obtained for stabilizability in the mean square sense for this setup \cite{7,14}. This bound relates the eigenvalues of the open loop plant to the minimal signal-to-noise ratio of the AWGN channel that is required for mean squared stabilizability. In \cite{72}, a similar bound when the objective is almost sure asymptotically stable was presented. This work has been extended to more general setups with sufficient conditions for stabilizability when sensor-controller communication is across a Gaussian product channel \cite{35}, a Gaussian relay channel \cite{34}, a Gaussian MAC and broadcast channel \cite{83}, a first order moving average Gaussian channel \cite{50} now known. Results on a network of point to point multi-hop Gaussian relay channels are also obtained in \cite{84}. Besides stabilizability, transient performance
and LQG and performance over the Gaussian channels are also discussed in [16, 17], respectively.

In this chapter, we consider the case when two processes need to be controlled with the sensor-controller communication occurring across a shared Gaussian multiple access (MAC) channel. A MAC channel has two transmitters with individual encoders. There is an individual average power constraint for each encoder. The channel output is the sum of the two channel inputs and an additive white Gaussian noise. Moving from an AWGN point to point channel to a MAC channel raises new challenges. It is known that stabilizing across an AWGN point to point channel can be done using an encoding and decoding algorithm that uses the classic Schalkwijk-Kailath (S-K) scheme [68] to transmit information across the AWGN channel at capacity. An S-K like scheme has been proposed by Ozarow [60] to achieve capacity across the MAC channel. However, the scheme superimposes block coding on top of the S-K scheme and hence cannot be directly used for control [66]. Since there are two processes being controlled simultaneously, some aspects of distributed control become important. While distributed control has been considered in the presence of communication channels [42, 55, 82], typically these works assume the communication network to be a collection of point to point channels. A MAC channel has the feature that transmission from one source interferes with the transmission from the other source. Thus, these works are not immediately applicable. While one obvious possibility is to perform some scheduling for the two sources, it is known that such a strategy leads to rate loss [60] if the objective is to transmit data at the maximum rate possible.

We characterize the region in the plane defined by the rates at which the two encoders transmit that is required for stabilizability of the two processes across a MAC channel. We consider two cases: one in which the two encoders can cooperate by exchanging information and another in which no information is exchanged between
the encoders. We provide necessary and sufficient conditions for stabilizability that provide inner and outer bounds on the rate region required for stabilizability. In particular, for the case with information exchange, the two bounds coincide. On the other hand, if no information is being exchanged, there is still a gap between the two bounds. In this case, we obtain an inner bound for the sufficient rate region. We consider both the case when there is no process noise and when there is process noise. The sufficient region is shown to be smaller in the case with process noise comparing to the noiseless case. In the noisy case, interestingly, the sufficient rate region is larger for the case when two process noises are correlated with each other than when they are not.

As was pointed out in [66], the Shannon capacity of a communication channel may not be sufficient to characterize the rate required to stabilize a control system across it in the mean squared sense. Rather, the notion of anytime capacity, which is, in general, less than the Shannon capacity, is needed. For control across a point to point AWGN channel, the two capacities are equal and the Shannon capacity characterization of the AWGN channel suffices to provide necessary and sufficient conditions for stabilizability. For the case when the encoders exchange information, our results, thus, indicate that the anytime capacity region of the MAC channel is the same as the Shannon capacity region. For the case without information exchange, we obtain necessary conditions for stabilizability by considering the Shannon capacity region. However, the fact that there is a gap between the necessary and sufficient conditions for stabilizability leads to the conjecture that anytime capacity in this setting may not be the same as the Shannon capacity.

The work closest to ours is [83] which also considered the problem of stabilizability of two plants across a MAC channel. [83] considers a specific point in the rate region.

\footnote{In this case, the sufficient conditions that we provide include the conditions in [83] as a special case.}
and proves that it is sufficient for stabilizability. Since the point corresponds to the case when Ozarow’s scheme does not involve a block code, the design of encoders and decoders in [83] can use Ozarow’s scheme directly. We provide more points in the rate region that are sufficient for stabilizability. For these points, Ozarow’s scheme has a block code component and thus cannot be used directly. We also provide necessary conditions for stabilizability and the case when the encoders are able to cooperate. More importantly, we consider the problem in the presence of process noise, so that the stabilization problem is no longer limited to transmitting information about the initial condition alone. This case leads to interesting insights on the use of signaling to coordinate between the two encoders if the process noises for the two plants are correlated. A preliminary version of some of these results are also presented in [43].

The rest of the chapter is organized as follows. Section 4.2 formulates the problem and presents the main results. Section 3.3 provides the proofs for the results in the case when there is no information exchange between the encoders. Section 3.3.1 proves the necessary condition of the stabilizability in the no information exchange case. Section 3.3.2 proves the sufficient condition for stabilizing systems, where the case without process noise is discussed in 3.3.2.1 and the noisy case is discussed in 3.3.2.2. Section 3.4 proves the results under the assumption that the encoders exchange information. Section 5.6 concludes the chapter.

Notations: $| \cdot |$ denotes the magnitude of a complex number. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{E}$ denotes the expectation operator. For a variable $x_{i,t}$ with two subscripts, the first subscript denotes the plant and the second denotes the time index. If not needed for a variable, we omit the plant index and write the variable merely as $x_t$. A sequence $X_{k_1}, \ldots, X_{k_2}$ is written as $X_{k_1}^{k_2}$ and if $k_1 = 0$, as $X^{k_2}$. $\text{sgn}(\delta)$ denotes the sign function. $\text{sgn}(\delta) = 1$ if $\delta \geq 0$ and $\text{sgn}(\delta) = -1$ if $\delta < 0$. 

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3.2 Problem Statement and Statement of Main Results

Consider two linear scalar plants that evolve as

\[ x_{i,t+1} = \lambda_i x_{i,t} + u_{i,t} + w_{i,t} \quad \text{for} \quad i = 1, 2, \]  

(3.1)

where \( x_{i,t} \in \mathbb{R} \) is the state of the \( i \)-th plant at time \( t \) and \( u_{i,t} \in \mathbb{R} \) is the corresponding control input. The eigenvalues are assumed to be unstable. Thus, \(|\lambda_i| > 1\). We assume that the initial values \( x_{i,0} \) are mutually independent random variables with zero mean and variance \( \sigma^2_{i,0} \). For each plant, the process noise \( w_{i,t} \) is assumed to be white, i.i.d. Gaussian with mean zero, variance \( \sigma^2_{w,i} \) and cross correlation coefficient \( \gamma_t = \frac{\mathbb{E}w_{1,t}w_{2,t}}{\sqrt{\sigma^2_{w,1}\sigma^2_{w,2}}} \). The initial values \( x_{i,0}, i = 1, 2 \) and the process noises \( w_{i,t}, i = 1, 2 \) are assumed to be mutually independent.

As illustrated in Figure 3.1, the state of each plant is encoded and transmitted across a shared MAC channel. In particular, the sensor \( i \) is assumed to observe the exact value of state \( x_{i,t} \) at each step \( t \). This state value is encoded into the value \( X_{i,t} = f_i(t, x_{i,0}, \ldots, x_{i,t}) \) and transmitted. We impose an average power constraint of the form \( \lim_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T} \mathbb{E}X_{i,t}^2 \leq P_i \).
The channel output is given by

\[ Y_t = X_{1,t} + X_{2,t} + Z_t, \]  

(3.2)

with zero mean white Gaussian noise \( Z_t \sim \mathcal{N}(0, N) \) that is independent of the other random variables in the system. This output is observed by the two decoders. Since the decoders are collocated with each other and with the controller, the decoders and the controllers can be considered to be effectively a single decoder or controller. The controller computes and transmits two control inputs \( u_{i,t} = g_i(t, Y^t) \). The control input \( u_{i,t}, i = 1, 2 \) is applied to plant \( i \). For such a system, we define stability as follows.

**Definition 1. (Mean Square Stability)** The system \( (3.1) \) is stable in the mean square sense if and only if

\[ \limsup_{t \to \infty} \mathbb{E}[x_{i,t}^2] < c, \quad i = 1, 2. \]

(3.3)

where \( c \) is a finite positive number. If there exist design parameters \( f_i(\cdot), g_i(\cdot) \), for \( i = 1, 2 \) such that the above equation holds, the system is said to be stabilizable.

For this set-up, we consider two cases. In the first case, called no information exchange, the encoders design functions of the form \( X_{i,t} = f_i(t, x_i^t) \) as explained above. Thus, the transmission from each plant is a function only of its own data. In the second case, called information exchange case, the encoders design functions are assumed to be of the form \( X_{i,t} = f_i(t, x_1^t, x_2^t) \). In other words, the encoders can exchange the information they have access to and jointly encode the two state values.

The following are the main results of the chapter.

**Theorem 3.2.1.** Consider the problem formulated above for the no information exchange case. If the system \( (3.1) \) is stabilizable in the mean square sense through any
choice of the design parameter $f_i(\cdot), g(\cdot)$, the following inequalities must be satisfied,

$$
\log |\lambda_1| \leq \frac{1}{2} \log \left(1 + \frac{P_1(1 - \rho^2)}{N}\right), \\
\log |\lambda_2| \leq \frac{1}{2} \log \left(1 + \frac{P_2(1 - \rho^2)}{N}\right), \\
\log |\lambda_1| + \log |\lambda_2| \leq \frac{1}{2} \log \left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N}\right), 
$$

(3.4)

for some $\rho$ such that $0 \leq \rho \leq 1$.

Proof. See Section 3.3.1

Theorem 3.2.2. The system (3.1) without process noise, i.e. $w_{i,t} = 0, i = 1, 2$ and in the no information exchange case, is stabilizable in the mean square sense if

$$
\log |\lambda_1| < \frac{1}{2} \log \left[\left(1 + \frac{P_1}{N}\right)^{\frac{M-1}{M}} \left(1 + \frac{P_1(1 - \rho^2)}{N}\right)^{\frac{1}{M}}\right], \\
\log |\lambda_2| < \frac{1}{2} \log \left(1 + \frac{P_2(1 - \rho^2)}{N}\right)^{\frac{1}{M}}, 
$$

(3.5)

for any $M \geq 1, M \in \mathbb{N}$, where the parameter $\rho'$ is a root of the equation

$$
\rho^2 \left(\frac{N + P_1}{N}\right)^{M-1} - 1 = \frac{N(N + 2|\rho|\sqrt{P_1 P_2} + P_1 + P_2)}{(N + P_1(1 - \rho^2))(N + P_2(1 - \rho^2))}.
$$

Proof. See Section 3.3.2.1

Remark 3.2.3. Ozarow [60] proved that for the Gaussian MAC channel with no information exchange between the encoders, the achievable rate region is given by

$$
\bigcup_{0 \leq \rho \leq 1} \left\{ (R_1, R_2) : 0 \leq R_1 \leq \frac{1}{2} \log \left(1 + \frac{P_1}{N}(1 - \rho^2)\right), \\
0 \leq R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_2}{N}(1 - \rho^2)\right), \\
0 \leq R_1 + R_2 \leq \frac{1}{2} \log \left(1 + \frac{P_1 + P_2 + 2\rho\sqrt{P_1 P_2}}{N}\right) \right\},
$$

(3.6)
If we use the data rate theorem [73] which implies that a system of the form (3.1) can be stabilized across a digital wireless channel only if the data at the plant is transmitted at a rate $R$ to the controllers such that $R > \log |\lambda|$, then the region (3.4) can be interpreted as an extension of the necessity of the data rate theorem to the MAC setting. Since there is a gap between the sufficient condition in (3.5) and the necessity condition in (3.4), the sufficiency of the data rate theorem may not extend in a similar manner to the MAC setting.

Remark 3.2.4. The condition in [14] includes, in particular, the point proved to be sufficient for stability in [83]. This is the only point for which the regions described by (3.5) and (3.6) intersect and is given by the value $\rho = \rho^*$, where $\rho = \rho^*$ is the solution of the nonlinear equation

$$
(N + P_1(1 - \rho^2))(N + P_2(1 - \rho^2)) = N(N + 2|\rho|\sqrt{P_1P_2} + P_1 + P_2). \tag{3.7}
$$

This point is obtained if we set $M = 1$ in (3.5).

For the case when the system (3.1) has process noises, a sufficient condition for stabilizability is given in the following.

**Theorem 3.2.5.** Consider the system (3.1) in the no information exchange case and suppose that the process noises $w_{1,t}$ and $w_{2,t}$ satisfy $E[w_{1,t}w_{2,t'}] = c\delta(t - t')$, $c > 0$, $t \neq t'$ with the correlation coefficient $\gamma_t$ being

$$
\gamma_t = \frac{c}{\sqrt{\sigma^2_{w_1}\sigma^2_{w_2}}} = \gamma. \tag{3.8}
$$

The system with correlated process noise is stabilizable only if (3.4) holds. For
the sufficient part, if $\lambda_1\lambda_2 < 0$, then a sufficient condition for its stabilizability is

$$\log |\lambda_1| < \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2|\rho''|\sqrt{P_1P_2} + N}{N + P_2(1 - (\rho'')^2)} \right), \quad (3.9)$$

$$\log |\lambda_2| < \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2|\rho''|\sqrt{P_1P_2} + N}{N + P_1(1 - (\rho'')^2)} \right), \quad (3.10)$$

where $\rho''$ is the solution of the following nonlinear equation,

$$\rho = \frac{\lambda_1\lambda_2(N\rho - \text{sgn}(\rho)\sqrt{P_1P_2}(1 - \rho^2))}{P_1 + P_2 + 2|\rho|\sqrt{P_1P_2} + N} + \gamma\sqrt{1 - \lambda_1^2\beta_1(\rho)}(1 - \lambda_2^2\beta_2(\rho)) \quad (3.11)$$

and $\beta_i(\rho), i = 1, 2$ is defined as

$$\beta_1(\rho) = \frac{N + P_2(1 - \rho^2)}{P_1 + P_2 + 2|\rho|\sqrt{P_1P_2} + N}, \quad (3.12)$$

$$\beta_2(\rho) = \frac{N + P_1(1 - \rho^2)}{P_1 + P_2 + 2|\rho|\sqrt{P_1P_2} + N}. \quad (3.13)$$

**Proof.** See Section 3.3.2.2

**Theorem 3.2.6.** Consider the problem formulated above, for the case when information exchange is allowed between the two encoders. The system (3.1) is stabilizable in the mean square sense if and only if

$$\log |\lambda_1| + \log |\lambda_2| \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1P_2}}{N} \right). \quad (3.14)$$

**Proof.** See Section 3.4

**Remark 3.2.7.** This region described by (3.14) coincides with the sum rate bound in (3.4) with $\rho = 1$. This is because the information exchange allows the cooperation of two plants to transmit the feedback information of both states using the power of both transmitters.
3.3 Stabilizability Proof for System with No Information Exchange

3.3.1 Proof for Necessary Condition of Stabilizability

In this section, we prove the necessary condition for stabilizability as summarized in Theorem 3.2.1. We adopt an information theoretic approach. Preliminary concept from information theory is also stated in Appendix 3.6.1.

We first bound the directed mutual information between the channel input and output in terms of the Gaussian MAC channel capacity. Then, we relate the system parameters with the direct mutual information. We begin with the following initial results.

**Lemma 3.3.1.** For the system (3.1) described in Section 4.2, the channel input sequences $X_1^{T-1}, X_2^{T-1}$ and channel output $Y^{T-1}$ satisfy the following inequalities.

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_t^1; Y_t | Y_{t-1}^{t-1}, X_t^2) \leq \frac{1}{2} \log \left( 1 + \frac{P_1(1 - \rho^2)}{N} \right),
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_t^2; Y_t | Y_{t-1}^{t-1}, X_t^1) \leq \frac{1}{2} \log \left( 1 + \frac{P_2(1 - \rho^2)}{N} \right),
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_t^1, X_t^2; Y_t | Y_{t-1}^{t-1}) \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\rho \sqrt{P_1P_2}}{N} \right). \tag{3.15}
\]

A similar result can be found in the Gaussian point-to-point channel in [14], where the relationship between directed mutual information and the Gaussian channel capacity is established. Directed information is required to take into account the casualty between the channel input and channel output as sequences generated by the LTI feedback system.

**Proof.** See Appendix.

**Lemma 3.3.2.** For the system (3.1) described in Section 4.2, the directed information between channel input sequences $X_1^{T-1}, X_2^{T-1}$ and channel output $Y^{T-1}$ and the
mutual information between the state $x_{i,t}, i = 1, 2$ and channel output satisfies the following relationship.

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X^t_1; Y^t | Y^{t-1}, X^t_2) \geq \frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}; Y^t | Y^{t-1}, x_{2,t}), \quad (3.16)
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X^t_2; Y^t | Y^{t-1}, X^t_1) \geq \frac{1}{T} \sum_{t=0}^{T-1} I(x_{2,t}; Y^t | Y^{t-1}, x_{1,t}), \quad (3.17)
\]

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X^t_1, X^t_2; Y^t | Y^{t-1}) \geq \frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}, x_{2,t}; Y^t | Y^{t-1}). \quad (3.18)
\]

**Proof.** See Appendix.

**Lemma 3.3.3.** For the problem setting described in Section 4.2, if the system (3.1) is mean square stable, then the following inequalities hold,

\[
\liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}; Y^t | Y^{t-1}, x_{2,t}) \geq \log |\lambda_1|, \quad (3.19)
\]

\[
\liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} I(x_{2,t}; Y^t | Y^{t-1}, x_{2,t}) \geq \log |\lambda_2|, \quad (3.20)
\]

\[
\liminf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}, x_{2,t}; Y^t | Y^{t-1}) \geq \log |\lambda_1| + \log |\lambda_2|. \quad (3.21)
\]

**Proof.** See Appendix.

Now, we are ready to prove the necessary condition of the noisy system, given that the process noise are independent from each other. [Proof of the necessary condition (3.4)]

**Proof.** Combining Lemma 3.3.1 and Lemma 3.3.2, we can establish the inequalities of mutual informations between the states and channel output and the Gaussian MAC.
channel capacity region, given as

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}; Y_t | Y_{t-1}, x_{2,t}) \leq \frac{1}{2} \log \left( 1 + \frac{P_1(1 - \rho^2)}{N} \right),
\] (3.22)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(x_{2,t}; Y_t | Y_{t-1}, x_{1,t}) \leq \frac{1}{2} \log \left( 1 + \frac{P_2(1 - \rho^2)}{N} \right),
\] (3.23)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(x_{1,t}, x_{2,t}; Y_t | Y_{t-1}) \leq \left( 1 + \frac{P_1 + P_2 + 2\rho \sqrt{P_1 P_2}}{N} \right).
\] (3.24)

By Lemma 3.3.3 we established the inequalities of mutual informations between the states and channel output and the logarithms of eigenvalues of the two LTI systems, which is the inequalities (3.19)(3.20)(3.21). Combining these inequalities and the inequalities (3.22)(3.23)(3.24), by taking limit infimum with respect to \( T \to \infty \), we obtain the necessary condition (3.4).

3.3.2 Proof of Sufficient Condition of Stabilizability

3.3.2.1 Proof of Sufficient Condition of Stabilizability of Noiseless Systems

We now prove the sufficient condition for stabilizability of system (3.1), with \( w_{i,t} = 0, i = 1, 2 \), as stated in Theorem 3.2.2. As discussed earlier, there is no clear analogy of an S-K like scheme for MAC channels. Although Ozarow [60] has provided a similar scheme for transmitting at capacity across MAC channels, his scheme has two components. In the first component, the two encoders implement an S-K scheme like scheme and in the second component, the encoders implement a block code and while decoder implements successive decoding. The latter component is required to achieve all points in (3.4) except the case when \( \rho = \rho^* \). Since block codes are known to be insufficient to allow moment stability an LTI plant [67], Ozarow’s scheme cannot be directly used in our setup. Whether the rate region described by (3.4) is sufficient for stabilizability is, thus, an open question. We present a particular encoder-decoder design to provide an inner bound to the rate region sufficient for stability.
We would like to point out that \[83\], the same problem was considered. However, that work focused on the case when $\rho = \rho^*$ in which case Ozarow’s scheme does not have the block coding component and can be directly used for stabilization. We extend this region by considering other points beyond $\rho = \rho^*$, when Ozarow’s scheme cannot be directly used.

**Proof of Theorem 3.2.2.** Although the initial condition $x_{i,0}, i = 1, 2$ is arbitrarily distributed, the states $x_{i,2}, i = 1, 2$ can be converted to a Gaussian distribution by using the method in \[83\]. Thus, for notational ease and without loss of generality, we assume that $x_{i,0}, i = 1, 2$ is distributed according to a Gaussian distribution.

**Control and Communication Scheme**

Our control and communication scheme is periodic with a period of $M$ steps, where $M$ is a parameter to be chosen by design. The aim of the scheme is to allow the controller to calculate the MMSE estimates $\hat{x}_{i,t}$ of the states $x_{i,t}, i = 1, 2$ at every time $t$. The estimates $\hat{x}_{i,t}, i = 1, 2$ are then used to calculate the control input. We choose a minimum variance control law $u_{i,t} = -\lambda_i \hat{x}_{i,t}$, for any $t \geq 0$. Under this control law, the state variances $\alpha_{i,t} = \mathbb{E}x_{i,t}^2, i = 1, 2$ evolve as

$$\begin{align*}
x_{i,t+1} &= \lambda_i x_{i,t} + u_{i,t}, \\
\alpha_{i,t+1} &= \lambda_i^2 \mathbb{E}(x_{i,t} - \hat{x}_{i,t})^2.
\end{align*} \tag{3.25}$$

We now describe the encoder transmissions and how the controller calculates $\hat{x}_{i,t}$.

At the $kM$-th step, $k \geq 0$, both transmitters transmit according to Ozarow’s
Thus, at these time steps, the channel inputs and output are given by

\[ X_{1,kM} = \sqrt{\frac{P_1}{\alpha_{1,kM}}} x_{1,kM}, \]
\[ X_{2,kM} = \sqrt{\frac{P_2}{\alpha_{2,kM}}} x_{2,kM} \text{sgn}(\rho_{kM}), \]
\[ Y_{kM} = X_{1,kM} + X_{2,kM} + Z_{kM}, \]

where \( \rho_{kM} = \frac{\mathbb{E}x_{1,kM}x_{2,kM}}{\sqrt{\alpha_{1,kM}\alpha_{2,kM}}} \) is the correlation coefficient of the two channel inputs. The function \( \text{sgn} \) is the signum function with \( \text{sgn}(\rho) = 1 \) for \( \rho \) non-negative and \( \text{sgn}(\rho) = -1 \) for \( \rho \) negative. At the decoder, the MMSE estimates of \( x_{i,kM} \), denoted by \( \hat{x}_{i,kM} \), can be obtained by computing the following quantities.

\[ \mathbb{E}x_{1,kM}Y_{kM} = \sqrt{\alpha_{1,kM}(\sqrt{P_1} + \sqrt{P_2}|\rho_{kM}|)}, \]  
\( (3.26) \)
\[ \mathbb{E}x_{2,kM}Y_{kM} = \sqrt{\alpha_{2,kM}(\sqrt{P_2} + \sqrt{P_1}|\rho_{kM}|)}\text{sgn}(\rho_{kM}), \]  
\( (3.27) \)
\[ \mathbb{E}Y_{kM}^2 = P_1 + P_2 + 2|\rho_{kM}|\sqrt{P_1P_2} + N, \]  
\( (3.28) \)
\[ \hat{x}_{i,kM} = \frac{\mathbb{E}x_{i,kM}Y_{kM}}{\mathbb{E}Y_{kM}^2}Y_{kM}, \]  
\( (3.29) \)

For the \( M - 1 \) steps within the same period, \( t \in [kM + 1, (k + 1)M - 1] k \geq 0 \), transmitter 1 transmits information about state \( x_1 \) and transmitter 2 keeps silent, \textit{i.e.}

\[ X_{1,t} = \sqrt{\frac{P_1}{\alpha_{1,t}}} x_{1,t}, \quad X_{2,t} = 0, \]  
\( (3.30) \)
\[ Y_t = \sqrt{\frac{P_1}{\alpha_{1,t}}} x_{1,t} + Z_t. \]  
\( (3.31) \)

These equations require calculation of the quantities \( \alpha_{i,t} \) and \( \rho_{kM} \). We now analyze the evolution of the state variances and the coefficient.

\textbf{Analysis of State Variances and Correlation Coefficient}
At the first step of each period, i.e., \( t = kM \), using (3.25) and (3.26) — (3.29), the state variances evolve as

\[
\begin{align*}
\alpha_{1,kM+1} &= \alpha_{1,kM} \lambda_1^2 \left( \frac{N + P_2(1 - \rho_{kM}^2)}{P_1 + P_2 + 2|\rho_{kM}| \sqrt{P_1 P_2} + N} \right), \\
\alpha_{2,kM+1} &= \alpha_{2,kM} \lambda_2^2 \left( \frac{N + P_1(1 - \rho_{kM}^2)}{P_1 + P_2 + 2|\rho_{kM}| \sqrt{P_1 P_2} + N} \right).
\end{align*}
\]  

(3.32)

During \( t \in [kM + 1, (k + 1)M - 1] \), the state variances evolve as follows.

\[
\begin{align*}
\alpha_{1,t+1} &= \alpha_{1,t} \lambda_1^2 \frac{N}{P_1 + N}, \\
\alpha_{2,t+1} &= \alpha_{2,t} \lambda_2^2.
\end{align*}
\]  

(3.33)

Thus, the state variances evolve over a complete period from \( kM \) to \( (k + 1)M \) as

\[
\begin{align*}
\alpha_{1,(k+1)M} &= \alpha_{1,kM} \lambda_1^{2M} \left( \frac{N}{P_1 + N} \right)^{M-1} \left( \frac{N + P_2(1 - \rho_{kM}^2)}{P_1 + P_2 + 2|\rho_{kM}| \sqrt{P_1 P_2} + N} \right), \\
\alpha_{2,(k+1)M} &= \alpha_{2,kM} \lambda_2^{2M} \left( \frac{N + P_1(1 - \rho_{kM}^2)}{P_1 + P_2 + 2|\rho_{kM}| \sqrt{P_1 P_2} + N} \right).
\end{align*}
\]  

(3.34)

The equations (3.32)—(3.34) can be used to calculate \( \alpha_{i,t} \) for use in (3.26), (3.27) and (3.30) starting with the initial condition \( \alpha_{i,0} \).

The correlation coefficient \( \rho_{kM} \) for use in (3.32) evolves as follows. First, we note that

\[
\mathbb{E}x_{1,(k+1)M} x_{2,(k+1)M} = \lambda_1^{M-1} \lambda_2^{M-1} \left( \frac{N}{P_1 + N} \right)^{M-1} \mathbb{E}x_{1,kM+1} x_{2,kM+1}.
\]

Using (3.26) — (3.29), The correlation \( \mathbb{E}x_{1,kM+1} x_{2,kM+1} \) is obtained as

\[
\mathbb{E}x_{1,kM+1} x_{2,kM+1} = \lambda_1 \lambda_2 \sqrt{\alpha_{1,kM} \alpha_{2,kM}} \cdot \left( \frac{N \rho_{kM} - \text{sgn}(\rho_{kM}) \sqrt{P_1 P_2}(1 - \rho_{kM}^2)}{P_1 + P_2 + 2|\rho_{kM}| \sqrt{P_1 P_2} + N} \right)
\]  

(3.35)

Then the coefficient \( \rho_{(k+1)M} \) at the end of the \( k \)-th period can be obtained using
\[
\rho_{(k+1)M} = \frac{\mathbb{E}x_{1,(k+1)M}x_{2,(k+1)M}}{\sqrt{\alpha_{1,(k+1)M}\alpha_{2,(k+1)M}}} = \lambda_1^M \lambda_2^M \left( \frac{N}{P_1 + N} \right)^{M-1} \frac{\sqrt{\alpha_{1,kM}\alpha_{2,kM}}}{\sqrt{\alpha_{1,(k+1)M}\alpha_{2,(k+1)M}}} \left( \frac{N\rho_{kM} - \text{sgn}(\rho_{kM})\sqrt{P_1P_2(1 - \rho_{kM}^2)}}{P_1 + P_2 + 2|\rho_{kM}|\sqrt{P_1P_2 + N}} \right).
\]  
(3.36)

By substituting (3.34) into (3.36), the coefficient \(\rho_{(k+1)M}\) can be further written as

\[
\rho_{(k+1)M} = \left( \frac{N}{P_1 + N} \right)^{(M-1)/2} \frac{N\rho_{kM} - \text{sgn}(\rho_{kM})\sqrt{P_1P_2(1 - \rho_{kM}^2)}}{\sqrt{(N + P_1(1 - \rho_{kM}^2))(N + P_2(1 - \rho_{kM}^2))}}.
\]  
(3.37)

Equation (3.37) can be used to calculate \(\alpha_{i,t}\) starting with the initial condition.

This defines our encoding and control schemes. To study the conditions for means square stability, we consider the evolution of \(\rho_{kM}\) in (3.37) as \(k \to \infty\). Let \(\rho_{kM} = \rho_{(k+1)M} = \rho\) and taking square of (3.37) and subtract 1, we obtain the following nonlinear equation

\[
\rho^2 \left( \frac{N + P_1}{N} \right)^{M-1} - 1 = (\rho^2 - 1) \frac{N(N + 2|\rho|\sqrt{P_1P_2 + P_1 + P_2})}{(N + P_1(1 - \rho^2))(N + P_2(1 - \rho^2))}.
\]  
(3.38)

The equation (3.38) has a unique solution for \(|\rho| = \rho'\), where \(0 < \rho' < 1\). This can be seen from the fact that the left hand side is greater than the right hand side when \(|\rho| = 0\) and the left hand side is less than the right hand side when \(|\rho| = 1\).

Similar to [60], the coefficient \(\rho_{kM}\) actually ends up with the switching sequence \((-1)^k \rho'\) as \(k \to \infty\). However, such a convergence is difficult to prove [60]. To avoid the proof of convergence, instead we initialize the states such that \(\rho_t = \rho'\) at \(t = 0\).

The initialization goes as follows. At \(t = -1\), both decoders inject a particular
Gaussian variable $\nu \sim \mathcal{N}(0, \sigma^2_\nu)$ such that

$$
\rho_{1,0} = \frac{\mathbb{E}x_{1,0}x_{2,0}}{\sqrt{\alpha_{1,0}\alpha_{2,0}}} = \frac{\sigma^2_\nu}{\sqrt{(\lambda^2_1\sigma^2_{1,-1} + \sigma^2_\nu)(\lambda^2_2\sigma^2_{2,-1} + \sigma^2_\nu)}}.
$$

(3.39)

Note that $\rho_{1,0} = 0$ when $\sigma^2_\nu = 0$ and $\rho_{1,0} \to 1$ when $\sigma^2_\nu \to \infty$. Since $0 < \rho' < 1$, there exist a proper $\sigma^2_\nu$ such that $\rho_{1,0} = \rho'$. After the initialization, the coefficient $\rho_{kM} = \rho'$ for all period $k$. The ratio between the variance $\alpha_{1,kM}$ and $\alpha_{1,(k+1)M}$ also becomes a constant. Using (3.34) and (3.38), the ratio which can be written as

$$
\frac{\alpha_{1,(k+1)M}}{\alpha_{1,kM}} = \lambda^2_1 \left( \frac{N}{P_1 + N} \right)^{M-1} \frac{N}{N + P_1(1 - \rho'^2)} \cdot \frac{\rho^2 - 1}{\rho^2((N + P_1)/N)^{M-1} - 1}
$$

< $\lambda^2_1 \left( \frac{N}{P_1 + N} \right)^{M-1} \frac{N}{N + P_1(1 - \rho'^2)}$.

Therefore, the state variance $\alpha_{1,kM} \to 0$ as $k \to \infty$ if the following condition holds.

$$
\lambda^2_1 \left( \frac{N}{P_1 + N} \right)^{M+1} \left( \frac{N}{N + P_1(1 - \rho'^2)} \right)^{\frac{1}{M}} < 1.
$$

(3.40)

Similarly, using (3.34) and (3.38), we obtain the ratio $\alpha_{2,(k+1)M}/\alpha_{2,kM}$ as

$$
\frac{\alpha_{2,(k+1)M}}{\alpha_{2,kM}} = \frac{N + P_1(1 - \rho'^2)}{P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N} < \frac{N}{N + P_2(1 - \rho'^2)}.
$$

Thus, $\alpha_{2,kM} \to 0$ as $k \to \infty$ if the following condition holds.

$$
\lambda^2_2 \left( \frac{N}{N + P_2(1 - \rho'^2)} \right)^{\frac{1}{2M}} < 1.
$$

(3.41)

Taking logarithm of the inequality (3.41), we obtain the sufficient condition (3.5) for stabilizability.

\[\blacksquare\]

Remark 3.3.4. The condition in (3.5) has $M$ as a design parameter. As shown in
Figure 3.2. Achievable points in the capacity region

Figure 3.2 by choosing different values for $M$, we obtain a series of points in the rate region lying outside the region achieved in [83]. By symmetry, if we switch the role of encoder 1 and 2 in the scheme explained in (3.30), the triangle points lying on the upper-left, can also be achieved. Note that these points lie between Ozarow’s capacity bound (3.4) and the achievable region in [83].

3.3.2.2 Stabilizability Condition for Systems with Process Noise

Section 3.3.2.1 considered the case when there is no process noise. In that case, it is sufficient to transmit information about the initial conditions of the two plants to the controllers. Indeed, that is the basic idea behind stabilizing schemes based on Ozarow’s algorithm. In this section, we study the stabilization of the systems with both plants disturbed by process noises at every time step. In this case, it is not sufficient to transmit information about only the initial conditions to the controllers.
We provide proofs for the sufficient rate region by construction of the stabilizing algorithms.

Intuitively, unlike the stabilizing process of the noiseless case, the evolution of the correlation coefficient $\rho_t$ now depends on the state variance $\alpha_{i,t}$. It is not straightforward to analyze the convergence of the coefficient by taking the limit with $t \to \infty$ because it is not obvious, at this point, how the state variances $\alpha_{i,t}, i = 1, 2$ evolves. Thus, we need to apply an initialization algorithm to inject proper random variables to drive the system state to desirable distribution with designated correlation coefficient before the stabilization process starts.

Proof of Theorem 3.2.5. Similar to the proof of the Theorem 3.2.2, we assume without loss of generality that the initial states are distributed as a Gaussian pdf.

Initialization: There are two objectives for this initialization process. One is to drive the states to have a certain distribution such that the correlation coefficient between the two states to a certain desirable value. The second objective is to set the variances of the states to a desirable values such that $\alpha_{i,1} = \alpha_{i,2}, i = 1, 2$, i.e. the state variances of the time step 1 and 2 of each plant has an equal value. We first describe the steps of achieving the first objective during the initialization process.

1) It is ideal that we drives the correlation coefficient between the two states to a certain desirable value, denoted as $\rho''$. The value of $\rho''$ is given by the solution of the following nonlinear equation.

$$\rho = \lambda_1 \lambda_2 h(\rho) + \gamma \sqrt{(1 - \lambda_1^2 \beta_1(\rho))(1 - \lambda_2^2 \beta_2(\rho))}, \quad (3.42)$$

where $h(\cdot), \beta_1(\cdot)$ and $\beta_2(\cdot)$ are given by

$$h(\rho_t) \triangleq \frac{N \rho_t - \text{sgn}(\rho_t) \sqrt{P_1 P_2(1 - \rho_t^2)}}{P_1 + P_2 + 2|\rho_t|\sqrt{P_1 P_2} + N}, \quad (3.43)$$
\[
\beta_1(\rho) \triangleq \frac{N + P_2(1 - \rho^2)}{P_1 + P_2 + 2|\rho|\sqrt{P_1P_2} + N}, \tag{3.44}
\]
\[
\beta_2(\rho) \triangleq \frac{N + P_1(1 - \rho^2)}{P_1 + P_2 + 2|\rho|\sqrt{P_1P_2} + N}. \tag{3.45}
\]

Given \(\lambda_1\lambda_2 < 0\), this equation \((3.42)\) always has a unique solution. The proof of the existence of a solution can be seen in Proposition \[3.6.3\] in Appendix \[3.6.3\]. The solution to equation \((3.42)\) is the desirable value of correlation coefficient \(\rho_t\), which will be explained later in the communication and control schemes.

Suppose the initialization process starts at \(t = -K\) and \(K \geq 1\) is the number of time steps chosen by design. The aim is to drive the states to have a certain distribution at time \(t = 0\). To this end, the decoder generates the following Gaussian random variables.

The first set of random variables is Gaussian variable \(\eta \sim \mathcal{N}(0, \sigma^2_\eta)\), which is mutually independent from all other random variables such as the process noises and the initial conditions in the process. The random variable \(\eta\) is injected into both plants at only \(t = -1\).

At every \(t \geq 0\), the decoder \(i\) also injects i.i.d random variable \(\mu_{i,t} \sim \mathcal{N}(0, \sigma^2_\mu)\) into the plant \(i\), for \(i = 1, 2\). This random variable is also mutually independent from all other random variables in the process.

The values of \(K, \sigma^2_\eta, \sigma^2_\mu\) are chosen to satisfy the following equations.

\[
\lambda_1^2\alpha_{1,0} + \sigma^2_\eta = \frac{\lambda_1^2\beta_1(\rho'')}{1 - \lambda_1^2\beta_1(\rho'')} (\sigma^2_\mu + \sigma^2_{w,1}),
\]
\[
\lambda_2^2\alpha_{2,0} + \sigma^2_\eta = \frac{\lambda_2^2\beta_2(\rho'')}{1 - \lambda_2^2\beta_2(\rho'')} (\sigma^2_\mu + \sigma^2_{w,2}), \tag{3.46}
\]

and

\[
\rho'' = \frac{\sqrt{\sigma^2_\eta + \gamma\sqrt{\sigma^2_{w,1}\sigma^2_{w,2}}}}{\sqrt{(\lambda_1^2\alpha_{1,0}^2 + \sigma^2_\mu + \sigma^2_{w,1})(\lambda_2^2\alpha_{2,0}^2 + \sigma^2_\mu + \sigma^2_{w,2})}}. \tag{3.47}
\]
where the state variances \( \alpha_{i,0}, i = 1,2 \) at \( t = 0 \) are given by \( \alpha_{i,0} = \lambda_i^{2K} \sigma_{i,0}^2 + \sum_{l=0}^{K-1} \sigma_{w,l}^2 \lambda_i^{2l} \). The equation (3.46) is obtained from the state variance \( \alpha_{i,1}, i = 1,2 \) with the random variable \( \eta \) and \( \mu_{i,t} \). The equation (3.47) is obtained by computing the correlation coefficient of the states \( \rho_t \) at the time \( t = 1 \).

We can find the value of \( \sigma_{\eta}^2, \sigma_{\mu,i}^2, i = 1,2 \), as follows. If the sufficient condition (3.9) holds, which implies that \( 1 - \lambda_i^2 \beta_i (\rho'') > 0, i = 1,2 \). Thus, both sides of the equation (3.46) is positive. Then, the variance \( \sigma_{\mu,i}^2, i = 1,2 \) can be found in terms of \( \sigma_{\eta}^2 \) using (3.46), denoted as \( \sigma_{\mu,i}^2 = l_i (\sigma_{\eta}^2) \). Then, we substitute \( \sigma_{\mu,i}^2 = l_i (\sigma_{\eta}^2) \) into the right hand side of (3.47) and we obtain

\[
\sigma_{\eta}^2 + \rho_w \sqrt{\sigma_{w,1}^2 \sigma_{w,2}^2} \quad \frac{\sqrt{\lambda_1^2 \alpha_{1,0}^2 + l_1 (\sigma_{\eta}^2) + \sigma_{\eta}^2 (\lambda_2^2 \alpha_{2,0}^2 + l_2 (\sigma_{\eta}^2) + \sigma_{\eta}^2)}}{\sqrt{\lambda_1^2 \alpha_{1,0}^2 + l_1 (\sigma_{\eta}^2) + \sigma_{\eta}^2 (\lambda_2^2 \alpha_{2,0}^2 + l_2 (\sigma_{\eta}^2) + \sigma_{\eta}^2)}}.
\]

(3.48)

Next, we show that there exists a value of \( \sigma_{\eta}^2 \) such that (3.48) can be equal to any given value \( 0 < \rho'' < 1 \), i.e. the equation (3.47) with respect to \( \sigma_{\eta}^2 \) has a solution. Note that the value of (3.48) has a lower bound \( L = \frac{\rho_w \sqrt{\sigma_{w,1}^2 \sigma_{w,2}^2}}{\sqrt{\lambda_1^2 \alpha_{1,0}^2 \lambda_2^2 \alpha_{2,0}^2}} \) when \( \sigma_{\eta}^2 = 0 \). By selecting large enough \( K \), the \( \alpha_{i,0} \) can be large enough such that \( L < \rho'' \) for any \( 0 < \rho'' < 1 \). The value of (3.48) approaches 1 when \( \sigma_{\eta}^2 \rightarrow \infty \). Therefore, there exists a solution \( \sigma_{\eta}^2 \) to the equation (3.47). After the value of \( \sigma_{\eta}^2 \) is obtained, we can find the values of \( \sigma_{\mu,i}^2 = l_i (\sigma_{\eta}^2) \).

Since (3.47) is the coefficient of the two states \( x_{i,1}, i = 1,2 \), then this initialization can achieve the first objective that the drives the coefficient to be \( \rho'' \).

2) We show that the initialization achieves the second objective which is to drive the variances of the state to a desirable pair of values.

Using the values obtained for \( \sigma_{\eta}^2, \sigma_{\mu,i}^2, i = 1,2 \) as solutions to equation (3.46) and (3.47), we can check the evolution of the state variance from \( t = 0 \) to \( t = 1 \), which is
written as

\[
\alpha_{i,1} = \lambda_{i,1}^2 \alpha_{i,0}^2 + \sigma_{\eta}^2 + \sigma_{\mu,i}^2
\]

\[
= \frac{\sigma_{w,i}^2 + \sigma_{\mu,i}^2}{1 - \lambda_i^2 \beta_i (\rho''(\rho'))}. \tag{3.49}
\]

The right hand side of (3.49) is the desirable values for the state variances. Such a value is selected because once the state variances are driven to such a value and then we will have \( \alpha_{i,t} = \alpha_{i,1}, i = 1, 2, t > 1 \). This will be shown later in the following communication and control schemes.

After the initialization process, the next communication and control schemes are applied for \( t > 1 \).

**Communication and Control Schemes:**

Once again the controller uses the MMSE estimate \( \hat{x}_{1,t} \) of the plant state and implement minimum variance control \( u_{i,t} = -\lambda_i \hat{x}_{i,t} \). Under this control law, the state variance evolves as

\[
\alpha_{i,t+1} = \lambda_i^2 \mathbb{E} (x_{i,t} - \hat{x}_{i,t})^2 + \sigma_{w,t}^2 + \sigma_{\mu,t}^2. \tag{3.50}
\]

The encoder transmissions are as follows. At the \( t \)-th step, the transmitters transmit the input.

\[
X_{1,t} = \sqrt{\frac{P_1}{\alpha_{1,t}}} x_{1,t}, \quad X_{2,t} = \sqrt{\frac{P_2}{\alpha_{2,t}}} x_{2,t} \text{sgn}(\rho_t), \tag{3.51}
\]

and the channel output is given as,

\[
Y_t = \sqrt{\frac{P_1}{\alpha_{1,t}}} x_{1,t} + \sqrt{\frac{P_2}{\alpha_{2,t}}} x_{2,t} \text{sgn}(\rho_t) + Z_t, \tag{3.52}
\]

where \( \rho_t = \frac{\mathbb{E} x_{1,t} x_{2,t}}{\sqrt{\alpha_{1,t} \alpha_{2,t}}} \) is the correlation coefficient of the two states \( x_{i,t}, i = 1, 2 \). The
decoder computes the MMSE estimates of \( \hat{x}_{i,t} = \frac{E_{x_{i,t}}Y_t}{E_{Y_t}^2}Y_t \) using the relations

\[
E_{x_{1,t}}Y_t = \sqrt{\alpha_{1,t}}(\sqrt{P_1} + \sqrt{P_2} |\rho_t|),
\]
\[
E_{x_{2,t}}Y_t = \sqrt{\alpha_{2,t}}(\sqrt{P_2} + \sqrt{P_1} |\rho_t|) \text{sgn}(\rho_t),
\]
\[
E_{Y_t}^2 = P_1 + P_2 + 2|\rho_t|\sqrt{P_1 P_2} + N.
\] (3.53)

**Analysis of Evolution of State Variances and Correlation Coefficient:**

Next, we study the evolution of the state variances and correlation coefficient under the communication and control scheme.

By (3.50) and (3.53), the mean square of the estimation error can be computed as

\[
E(x_{i,t} - \hat{x}_{i,t})^2 = \beta_i(\rho_t)\alpha_{i,t}, i = 1, 2
\] (3.54)

Then, the state variance evolves as

\[
\alpha_{i,t+1} = \lambda_i^2 \beta_i(\rho_t)\alpha_{i,t} + \sigma_{w,i}^2 + \sigma_{\mu,i}^2, i = 1, 2.
\] (3.55)

The evolution of the coefficient \( \rho_t \) can be obtained as follows. The correlation

\[
E_{x_{1,t+1}x_{2,t+1}} = E[\lambda_1(x_{1,t} - \hat{x}_{1,t}) + w_{1,t}][\lambda_2(x_{2,t} - \hat{x}_{2,t}) + w_{2,t}]
\]
\[
= \lambda_1 \lambda_2 E[(x_{1,t} - \hat{x}_{1,t})(x_{2,t} - \hat{x}_{2,t})] + E_{w_{1,t}w_{2,t}}
\]
\[
= \lambda_1 \lambda_2 \sqrt{\alpha_{1,t}\alpha_{2,t}} \frac{N\rho_t - \text{sgn}(\rho_t)\sqrt{P_1 P_2}(1 - \rho_t^2)}{P_1 + P_2 + 2|\rho_t|\sqrt{P_1 P_2} + N} + E_{w_{1,t}w_{2,t}},
\]
where $E_{w_1,t}w_{2,t} = \gamma \sqrt{\sigma^2_{w_1} \sigma^2_{w_2}}$. The correlation coefficient $\rho_t$ evolves as

$$\rho_{t+1} = \frac{E_{x_{1,t+1}x_{2,t+1}}}{\sqrt{\alpha_{1,t+1}\alpha_{2,t+1}}} = \lambda_1 \lambda_2 \frac{N \rho_t - \text{sgn}(\rho_t)\sqrt{P_1P_2(1-\rho_t^2)}}{P_1 + P_2 + 2|\rho_t|\sqrt{P_1P_2} + N} + \gamma \sqrt{\sigma^2_{w_1} \sigma^2_{w_2}} \sqrt{\alpha_{1,t+1}\alpha_{2,t+1}}. \tag{3.56}$$

Using (3.49) and (3.54) and $\rho_t = \rho''$, the state variance at $t = 2$ satisfies $\alpha_{i,2} = \alpha_{i,1}$. Substituting $\rho_t = \rho''$ and $\alpha_{i,2} = \alpha_{i,1}, i = 1, 2$ into the evolution of the state variances (3.42) and the evolution of the coefficient $\rho_t$ (3.56), it can be seen that the coefficient $\rho_t = \rho''$ remains the same value for $t = 2$. By applying (3.55) and (3.56) successively for $t \geq 2$, we can see that the state variances and coefficient remains the same value at $t = 2$. In other words, the system reaches steady state at $t = 2$.

In summary, if the condition (3.9) holds, then the initialization algorithm ensures that the state variance reaches steady state $\alpha_{i,t} = \alpha_{i,1}$ under the control scheme described in (3.51) for $t \geq 1$. Also, the correlation coefficient $\rho_t$ remains the same value $\rho''$ for $t \geq 1$. Therefore, the sufficient condition for stabilizing the system with process noise is (3.9).

An interesting observation is achieved if we consider the special case with process noises uncorrelated, i.e. $\gamma = 0$.

**Corollary 1.** The system (3.1) with process noises present, no information exchange case and $\gamma = 0$ is stabilizable in the mean squared sense if the following inequalities
\[
\log |\lambda_1| < \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N}{N + P_2(1 - (\rho')^2)} \right),
\]
\[
\log |\lambda_2| < \frac{1}{2} \log \left( \frac{P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N}{N + P_1(1 - (\rho')^2)} \right),
\]

(3.57)

where \(|\rho'|\) is the solution to the following nonlinear equation

\[
\rho^2 = \left( \frac{\lambda_1 \lambda_2 (N\rho - \text{sgn}(\rho)\sqrt{P_1P_2}(1 - \rho^2))}{P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N} \right)^2.
\]

(3.58)

**Proof.** This result can be proved immediately by following the same initialization process and schemes of communication and control in the proof of Theorem 3.2.5. The existence of solution to the nonlinear equation (3.7) is proved in Appendix 3.6.3. \(\Box\)

**Lemma 3.3.5.** The value of the coefficient \(\rho'\) in Corollary 1 satisfies \(0 < \rho' < \rho^*\), where \(\rho^*\) is the solution to the nonlinear equation (3.7).

**Proof.** By Corollary 1, the coefficient \(\rho'\) satisfies the nonlinear equation (3.57), which is equivalent to

\[
\lambda_1 \lambda_2 < \frac{P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N}{\sqrt{(N + P_2(1 - (\rho')^2))(N + P_1(1 - (\rho')^2))}}.
\]

(3.59)

Substituting (3.59) into (3.58), we obtain the following inequality.

\[
|\rho'|^2 < \left( \frac{N\rho - \text{sgn}(\rho)\sqrt{P_1P_2}(1 - \rho^2))}{\sqrt{(N + P_2(1 - (\rho')^2))(N + P_1(1 - (\rho')^2))}} \right)^2.
\]

(3.60)

Subtracting 1 from both sides of (3.60) and then divide \(|\rho'|^2 - 1\) from both sides, we obtain

\[
\frac{N(P_1 + P_2 + 2|\rho'|\sqrt{P_1P_2} + N)}{(N + P_2(1 - (\rho')^2))(N + P_1(1 - (\rho')^2))} < 1
\]

(3.61)
Recall that
\[
\frac{N(P_1 + P_2 + 2|\rho^*|\sqrt{P_1 P_2} + N)}{(N + P_1(1 - (\rho^*)^2))(N + P_2(1 - (\rho^*)^2))} = 1
\]
then it can be seen that \(\rho' < \rho^*\).

Remark 3.3.6. One intuitive explanation to the fact that \(\rho' < \rho^*\) is that with uncorrelated process noise injected into the plant, the correlation \(\mathbb{E}x_{1,t}x_{2,t}\) remains the same but the state variances increase, which leads to a smaller coefficient \(\rho'\). In other words, mutually independent process noises \(w_{1,t}\) and \(w_{2,t}\) reduces the correlation between the states. But the correlation between the states is in fact helpful for communication over the MAC channel, which implies that with less correlation in the states, the point with \(\rho = \rho^*\) is no longer achievable.

The following result studies the relationship between the coefficient \(\rho'\) achieved with uncorrelated process noises and the coefficient \(\rho''\) achieved with correlates process noises.

Lemma 3.3.7. Consider the same system model with the process noises are uncorrelated and then according to (3.57), correlation coefficient can be found as \(\rho'\), then the correlation coefficient \(\rho'' > \rho'\) for \(\gamma > 0\).

Proof. Denote the right hand side of the nonlinear equation (3.42) as a function of the coefficient \(\rho\), which is written as \(\lambda_1 \lambda_2 h(\rho) + h_1(\rho)\), where \(h(\rho)\) is same as defined in (3.43) and \(h_1(\rho)\) is given by
\[
h_1(\rho) = \gamma \sqrt{(1 - \lambda_1^2 \beta_1(\rho))(1 - \lambda_2^2 \beta_2(\rho))},
\]
where \(\beta_i(\rho), i = 1, 2\) is defined in (3.54).

Consider the partial derivative \(\frac{\partial h(\rho)}{\partial \rho}\) for \(\rho > 0\), which is given by
\[
\frac{\partial h(\rho)}{\partial \rho} = \frac{(2\rho \sqrt{P_1 P_2} + N)(N + P_1 + P_2) + 2P_1 P_2(1 + \rho^2)}{(N + P_1 + P_2 + 2\rho \sqrt{P_1 P_2})^2} > 0.
\]

Thus, \(h(\rho)\) is increasing for \(\rho > 0\).
We show \( \rho' < \rho'' \) by contradiction. Suppose \( \rho' > \rho'' \). Since \( \rho' = \lambda_1 \lambda_2 h(\rho') \) and \( h_1(\rho'') > 0 \), the following inequality holds

\[
\rho' < \lambda_1 \lambda_2 h(\rho') + h_1(\rho''). \tag{3.64}
\]

Note that \( \lambda_1 \lambda_2 < 0 \) and \( h(\rho) \) is increasing in \( \rho \), the function \( \lambda_1 \lambda_2 h(\rho) \) is decreasing in \( \rho \). Given \( \rho' > \rho'' \), we have

\[
\lambda_1 \lambda_2 h(\rho') < \lambda_1 \lambda_2 h(\rho''). \tag{3.65}
\]

Combining (3.42), we obtain \( \rho' < \lambda_1 \lambda_2 h(\rho'') + h_1(\rho'') = \rho'' \), which contradicts the assumption \( \rho' > \rho'' \). Therefore, \( \rho' < \rho'' \).

**Remark 3.3.8.** By Lemma 3.3.7 it can be seen that the sufficient region of the system with \( \gamma > 0 \) is the outer bound of the region of the same system with uncorrelated process noise \( \gamma = 0 \). This is because \( \beta_i(\rho) \) is a decreasing function and thus \( \beta_i(\rho'') < \beta_i(\rho') \). This implies that the correlated process noise, actually helps the stabilization of the system over Gaussian MAC channels. One intuitive explanation is that the correlation of the process noise offers an information sharing opportunity to increase the sufficient region.

**Example 2.** Consider the system with transmission power constraints \( P_1 = 30 \) and \( P_2 = 30 \). The variance of the additive noise in the MAC channel is \( N = 5 \). The process noise variances \( \sigma^2_{w,1} = 20 \), \( \sigma^2_{w,2} = 30 \). The correlation coefficient of the process noises is \( \gamma = .5 \).

We select the value of each eigenvalue \( \lambda_1, \lambda_2 \) from 1 to 2.4 with increments of 0.01, and thus obtain \( 140 \times 140 \) eigenvalue pairs. For each pair, we check if the condition (3.9) is satisfied. Then, we can illustrate the sufficient region based on the approximate boundaries between the cluster of the points that satisfy the condition.
Figure 3.3. Sufficient Region for Systems with Process Noise

(3.9) and the cluster of points that violate that condition. The sufficient regions of both uncorrelated and correlated process noises are illustrated in Figure 3.3. Note that the sufficient region of the system with correlated noise is larger than the sum rate bound with $\rho = \rho^*$.

Remark 3.3.9. We can only prove that when $\lambda_1\lambda_2 < 0$, the equation (3.42) has a solution, which is Proposition 3.6.3 of Appendix 3.6.3. As is pointed in Remark 3.6.4 when $\lambda_1\lambda_2 > 0$, it is difficult to prove the existence of solution. In fact, the correlation coefficient $\rho_t$ does not converge to a specific point, which can be seen from the numerical result in Example 3. The coefficient $\rho_t$ results in a limit cycle.

In this case, the evolution of the correlation coefficient $\rho_t$ which is given in (3.56) depends on the state variances and it is difficult to calculate the values of $\rho_t$ at the limit cycle. That also means that we cannot do the same initialization as the case with $\lambda_1\lambda_2 < 0$ which drives the state variance to the steady state. Analytical results are difficult to obtain in this case.
Example 3. This example demonstrates the evolution of the system state variances \( \alpha_{i,t}, i = 1, 2 \) for the system with the same set of parameters as in Example 2, but with \( \lambda_1 \lambda_2 > 0 \). The eigenvalues \( \lambda_1 = 1.1, \lambda_2 = 1.2 \). The evolution of the state variances starts at the initial state variances are \( \sigma_1^2 = \sigma_2^2 = 5 \). We compute the evolution according to (3.50). The evolution of the correlation coefficient is illustrated in Figure 3.4. It can be seen that the correlation coefficient \( \rho_t \) becomes periodic in the steady state. Due to this behavior, the state variances oscillates as illustrated in Figure 3.5.

3.4 Stabilizability of Systems with Information Sharing

In this section, we prove the necessary and sufficient condition in Theorem 3.2.6 by a time sharing strategy.

Proof of Theorem 3.2.6. We first prove the sufficient condition by construction of the stabilizing algorithm. The control and communication schemes operate in a periodic manner with the period of \( M \) steps. In every period, the time interval
[kM, (k + 1)M − 1] is divided into two segments with length $m$ and $M − m$, where $m$ is a design parameter. Two different schemes are used separately in the two segments. For the $k$-th period, during $t \in [0, m − 1]$, both the transmitters transmit the state $x_{1,t}$. During the rest of the period $t \in [m, M − 1]$, both transmit the state $x_{2,t}$. Thus, for $i = 1, 2$, we have

$$X_{i,kM+t} = \begin{cases} \sqrt{\frac{P_i}{\alpha_{1,kM+t}}} x_{1,kM+t} & t \in [0, m - 1] \\ \sqrt{\frac{P_i}{\alpha_{2,kM+t}}} x_{2,kM+t} & t \in [m, M - 1]. \end{cases}$$

The channel output is

$$Y_t = \begin{cases} \sqrt{\frac{P_1}{\alpha_{1,t}}} x_{1,t} + \sqrt{\frac{P_2}{\alpha_{1,t}}} x_{1,t} + Z_t & t \in [0, m - 1] \\ \sqrt{\frac{P_1}{\alpha_{2,t}}} x_{2,t} + \sqrt{\frac{P_2}{\alpha_{2,t}}} x_{2,t} + Z_t & t \in [m, M - 1]. \end{cases}$$

During $t \in [kM, kM + m - 1]$, the decoder computes the MMSE estimate of $x_{1,t}$ and
during $t \in [kM + m, (k + 1)M - 1]$, it computes the MMSE estimate of $x_{2,t}$. The control input is once again calculated according to the minimum variance control law. Using (3.25), the variance of $x_{1,t+1}$ is thus given by

$$\alpha_{1,t+1} = \lambda_1^2 \alpha_{1,t} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right) + \sigma_{w,1}^2,$$

for $t \in [kM, kM + m - 1]$. Similarly, the variance of $x_{2,t+1}$ is thus given by

$$\alpha_{2,t+1} = \lambda_2^2 \alpha_{2,t} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right) + \sigma_{w,2}^2,$$

for $t \in [kM + m, (k + 1)M - 1]$. When the state of a plant is transmitted, that state of the other plant $x_{i,t}$ evolves open loop. Thus, over a complete period, the state variances evolve as

$$\alpha_{1,(k+1)M} = \alpha_{1,kM} \lambda_1^{2M} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right)^m + \sigma_{w,1}^2,$$

$$\alpha_{2,(k+1)M} = \alpha_{2,kM} \lambda_2^{2M} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right)^{M-m} + \sigma_{w,2}^2.$$ Clearly, the system is mean square stable if

$$\lambda_1^{2M} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right)^m < 1,$$

$$\lambda_2^{2M} \left( \frac{N}{P_1 + P_2 + 2\sqrt{P_1 P_2} + N} \right)^{M-m} < 1.$$ This leads to the condition below.

$$\log |\lambda_1| < \frac{1}{2} \frac{m}{M} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right),$$

$$\log |\lambda_2| < \frac{1}{2} \frac{M-m}{M} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right).$$
Adjusting the value \( m \) and \( M \), all of the points on the cooperative bound line can be achieved. Therefore, we proved the condition (3.14) in Theorem 3.2.6 is sufficient.

Next, we prove that the condition (3.14) is also necessary with the same argument in the necessity proof of Theorem 3.2.1 for the sum rate bound \( \log |\lambda_1| + \log |\lambda_2| \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right) \), which is the third inequality in (3.4). Note that in that proof, there is no restriction on the information exchange between the two transmitters, \( i.e. \) there is no requirement on the exact form of encoding function \( f_i \) and feedback control law. Thus, the arguments for proving the sum rate bound of (3.4) still holds for the case with information exchange. Thus, the third inequality in (3.4) still hold in this case. Moreover, since we already show that the sufficient region is the sum rate bound, then it is obvious that the individual bounds, \( i.e. \) the first two inequalities in (3.4) are not necessary.

Since the necessary sum rate bound satisfies the third inequality in (3.4), then we need to determine the coefficient \( \rho \). By the fact that the achievable region expands to the maximum value of \( \rho = 1 \), which is the maximum region characterized by the sum rate bound in (3.4), then we can conclude that \( \rho = 1 \) is associated with the necessary condition, which is given by (3.14) in Theorem 3.2.6, \( i.e. \)

\[
\log |\lambda_1| + \log |\lambda_2| < \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\sqrt{P_1 P_2}}{N} \right).
\]

3.5 Conclusions

In this chapter, we consider the problem of stabilizing two linear scalar plants over a Gaussian multiple access channel. The problem is studied with and without allowing sharing of information between both the encoders. Without information sharing, we obtain a necessary condition in terms of the SNR of the channel and the
magnitude of the eigenvalues. We also obtain a sufficient condition which expands the currently best known conditions. We consider both the cases when the system has process noises and when it does not. Under the assumption that full information sharing between the plants is allowed, the stabilizability condition becomes tight.

3.6 Appendix

3.6.1 Preliminaries

The proof the necessary condition relies on the information-theoretic arguments. A brief introduction of several important concepts is first given here.

Definition 2 (Differential Entropy). The differential entropy of a set of continuous random variables \(X_1, X_2, \ldots, X_n\) with density \(f(x_1, x_2, \ldots, x_n)\) is given as

\[
h(X) = -\int f(x^n) \log f(x^n) \, dx^n
\]  
(3.66)

We can also define the differential entropy of a random variable \(X\) conditioned on another random variable \(Y\). Both random variables have joint distribution \(f(x, y)\).

Definition 3 (conditional differential entropy).

\[
h(X|Y) = -\int f(x, y) \log f(x|y) \, dx \, dy
\]  
(3.67)

Definition 4 (Mutual Information). The mutual information of two random variables \(X, Y\) with joint density \(f(x, y)\) is defined as

\[
I(X; Y) = \int f(x, y) \log \frac{f(x, y)}{f(x)f(y)} \, dx \, dy
\]  
(3.68)

Moreover, the mutual information \(I(X; Y)\) can be represented in terms of condi-
tional entropy,

\[ I(X; Y) = h(X) - h(X|Y). \]  

(3.69)

3.6.2 Proofs of Technical Lemmas in Section 3.3.1

In this section of appendix, we prove Lemma 3.3.1 and the proof is based on the following lemma.

Lemma 3.6.1. For the system (3.1) described in Section 4.2, the following inequalities hold,

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}^t; Y_{t}^t|Y_{t-1}^t, X_{2,t}^t) \leq \frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}; Y_{t}|X_{2,t}), \]  

(3.70)

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X_{2,t}^t; Y_{t}^t|Y_{t-1}^t, X_{1,t}^t) \leq \frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}; Y_{t}|X_{2,t}), \]  

(3.71)

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}^t, X_{2,t}^t; Y_{t}^t|Y_{t-1}^t) \leq \frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}, X_{2,t}; Y_{t}). \]  

(3.72)

Proof. First, we prove (3.70), By definition, the conditioned \( I(X_{1,t}^t \rightarrow Y_{t}^t|Y_{t-1}^t) \) can be written as

\[ I(X_{1,t}^t \rightarrow Y_{t}^t|Y_{t-1}^t) = \sum_{t=0}^{T-1} I(X_{1,t}^t; Y_{t}|Y_{t-1}^t, X_{2,t}^t), \]  

(3.73)

where each mutual information on the right side satisfies

\[ I(X_{1,t}^t; Y_{t}|Y_{t-1}^t, X_{2,t}^t) = h(Y_{t}|Y_{t-1}^t, X_{2,t}^t) - h(Y_{t}|Y_{t-1}^t, X_{1,t}^t, X_{2,t}^t) \]

\[ \overset{(a)}{=} h(Y_{t}|Y_{t-1}^t, X_{2,t}^t) - h(Y_{t}|X_{1,t}, X_{2,t}) \]

\[ \overset{(b)}{=} h(Y_{t}|X_{2,t}) - h(Y_{t}|X_{1,t}, X_{2,t}) \]

\[ = I(X_{1,t}; Y_{t}|X_{2,t}). \]
The equality (a) holds because of the property of the discrete memoryless channel, the inequality (b) holds because conditioning does not increase the entropy. Substitute the inequality into (3.73), we obtain the inequality (3.70). By symmetry, the inequality (3.70) can also be obtained.

As for the third inequality (3.72), according to Theorem 2 in [48], we obtain

\[ \sum_{t=0}^{T-1} I(X^t_1, X^t_2; Y^t | Y^{t-1}) \leq \sum_{t=0}^{T-1} I(X^t_1, X^t_2; Y^t). \] (3.74)

Dividing both sides by \( T \), the inequality (3.72) can be proved. \( \square \)

Next, we introduce the following Lemma. The proof is borrowed from the necessity proof in [60], with a few adjustments for our problem.

**Lemma 3.6.2.** For the system (3.1) described in Section 4.2, the following inequalities hold,

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X^t_1, Y^t; X^t_2) \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{N} (1 - \rho^2) \right), \] (3.75)

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X^t_1, X^t_2; Y^t) \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{N} (1 - \rho^2) \right), \] (3.76)

\[ \frac{1}{T} \sum_{t=0}^{T-1} I(X^t_1, X^t_2, Y^t) \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\rho \sqrt{P_1 P_2}}{N} \right). \] (3.77)

**Proof.** Notice that

\[ I(X^t_1; Y^t | X^t_2) = h(Y^t | X^t_2) - h(Y^t | X^t_1, X^t_2), \] (3.78)

\[ I(X^t_1, X^t_2; Y^t) = h(Y^t) - h(Y^t | X^t_1, X^t_2). \] (3.79)

For any \( t \), since \( Y^t = X^t_1 + X^t_2 + Z^t \), we have

\[ h(Y^t | X^t_1, X^t_2) = h(Z^t) = \frac{1}{2} \log 2\pi e N. \] (3.80)
Let $\sigma_{i,t}^2 = \text{var}(X_i)$, and denote the covariance of $X_{1,t}$ and $X_{2,t}$ as $\mu_t$, then

$$h(Y_t) = h(X_{1,t} + X_{2,t} + Z_t)$$

$$\leq \frac{1}{2} \log 2\pi e \text{var}(X_{1,t} + X_{2,t} + Z_t)$$

$$= \frac{1}{2} \log 2\pi e (\sigma_{1,t}^2 + \sigma_{2,t}^2 + N + 2\mu_t).$$

Now assume that $X_{1,t} = x_1$. The conditional variance of $X_{2,t}$ given $X_{1,t}$ is no greater than the variance of $X_{2,t}$ around the linear estimate $\hat{x}_2 = (\mu/\sigma_1^2)x_1$. Then we have

$$\text{var}(X_{2,t}|X_{1,t}) := \mathbb{E}_{x_1} \text{var}(X_{2,t}|X_{1,t} = x_1)$$

$$\leq \mathbb{E}_{x_1} \mathbb{E}_{x_2|x_1} \left( X_{2,t} - \frac{\mu_t x_1}{\sigma_{1,t}^2} \right)^2$$

$$= \sigma_{2,t}^2 \left( 1 - \frac{\mu_t^2}{\sigma_{1,t}^2 \sigma_{2,t}^2} \right).$$

Therefore,

$$h(Y_t|X_{1,t}) = \mathbb{E}_{x_1} h(Y_t|X_{1,t} = x_1)$$

$$\leq \mathbb{E}_{x_1} \frac{1}{2} \log 2\pi e \text{var}(X_{1,t} + X_{2,t} + Z_t|X_{1,t} = x_1)$$

(3.81)

$$\leq \mathbb{E}_{x_1} \frac{1}{2} \log 2\pi e [\text{var}(X_{2,t}|X_{1,t} = x_1) + N],$$

where inequality (3.81) holds because $Z_t$ is independent of $X_{1,t}$. By Jensen’s inequality,

$$h(Y_t|X_{1,t}) \leq \frac{1}{2} \log 2\pi e [\mathbb{E}_{x_1} \text{var}(X_{2,t}|X_{1,t} = x_1) + N]$$

$$\leq \frac{1}{2} \log 2\pi e \left( \sigma_{2,t}^2 \left( 1 - \frac{\mu_t^2}{\sigma_{1,t}^2 \sigma_{2,t}^2} \right) + N \right).$$

Also we can obtain the similar bound for $h(Y_t|X_{2,t})$ by switching $\sigma_{1,t}^2$ and $\sigma_{2,t}^2$. Then
we have

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}; Y_t | X_{2,t}) \leq \frac{1}{T} \sum_{t=0}^{T-1} \frac{1}{2} \log \left[ \frac{\sigma_{1,t}^2 \left( 1 - \frac{\mu_t^2}{\sigma_{1,t}^2} \right) + N}{N} \right],
\]

(3.82)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{2,t}; Y_t | X_{1,t}) \leq \frac{1}{T} \sum_{t=0}^{T-1} \frac{1}{2} \log \left[ \frac{\sigma_{2,t}^2 \left( 1 - \frac{\mu_t^2}{\sigma_{2,t}^2} \right) + N}{N} \right],
\]

(3.83)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}, X_{2,t}; Y_t) \leq \frac{1}{T} \sum_{t=0}^{T-1} \frac{1}{2} \log \left[ \frac{\sigma_{1,t}^2 + \sigma_{2,t}^2 + 2\mu_t + N}{N} \right].
\]

(3.84)

Since the terms in the right side is concave in $\sigma_{1,t}^2, \sigma_{2,t}^2$ and $\mu_t$, so Jensen’s inequality can be used. Define

\[
\rho = \left( \frac{1}{T} \sum_{t=1}^{T-1} \mu_t \right) / \sigma_{1,t} \sigma_{2,t}.
\]

(3.85)

It is easy to see that $0 \leq \rho \leq 1$. Thus, as $t$ is large enough, we obtain

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}; Y_t | X_{2,t}) \leq \frac{1}{2} \log \left( 1 + \frac{P_1}{N} (1 - \rho^2) \right),
\]

(3.86)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{2,t}; Y_t | X_{1,t}) \leq \frac{1}{2} \log \left( 1 + \frac{P_2}{N} (1 - \rho^2) \right),
\]

(3.87)

\[
\frac{1}{T} \sum_{t=0}^{T-1} I(X_{1,t}, X_{2,t}; Y_t) \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\rho \sqrt{P_1 P_2}}{N} \right).
\]

(3.88)

Now, we can proceed to prove Lemma 3.3.1.

[Proof of Lemma 3.3.1]

Proof. Combining the inequalities in Lemma 3.6.1 and Lemma 3.6.2, the three inequalities in Lemma 3.3.1 hold.

[Proof of Lemma 3.3.2]
Proof. The inequality (3.16) can be obtained by considering the following mutual information.

\[
I((X^t_1; Y^t_1) \mid Y^{t-1} , X^t_2) = h(Y^t_1 \mid Y^{t-1} , X^t_2) - h(Y^t_1 \mid Y^{t-1} , X^t_1, X^t_2) \\
\overset{(a)}{=} h(Y^t_1 \mid Y^{t-1} , X^t_2) - h(Y^t_1 \mid Y^{t-1} , X^t_1, x_{1,t}, x_{2,t}) \\
\overset{(b)}{\geq} h(Y^t_1 \mid Y^{t-1} , x_{2,t}) - h(Y^t_1 \mid Y^{t-1} , x_{1,t}, x_{2,t}) \\
= I(x_{1,t}; Y^t_1 \mid Y^{t-1} , x_{2,t}),
\]

where (a) holds because the channel input and output and state \( x_{i,t}, i = 1, 2 \) forms a Markov chain \( Y_t \leftrightarrow X^t_1, X^t_2 \leftrightarrow x_{1,t}, x_{2,t} \), the inequality (b) holds because 1) conditioning does not increase the entropy 2) \( X_{2,t} \) is a function of \( x_{2,t} \) and the channel is memoryless, so we have \( h(Y^t_1 \mid Y^{t-1} , x_{2,t}) = h(Y^t_1 \mid Y^{t-1} , X^t_1, x_{2,t}) < h(Y^t_1 \mid Y^{t-1} , X^t_2) \).

By symmetry, the inequality (3.19) is also proved.

Next, we prove the inequality (3.18). For any \( 0 < t \leq T - 1 \),

\[
I((X^t_1, X^t_2); Y^t_1) = h(Y^t_1 \mid Y^{t-1}) - h(Y^t_1 \mid Y^{t-1} , X^t_1, X^t_2) \\
\overset{(a)}{=} h(Y^t_1 \mid Y^{t-1}) - h(Y^t_1 \mid Y^{t-1} , X^t_1, X^t_2, x_{1,t}, x_{2,t}) \\
\overset{(b)}{\geq} h(Y^t_1 \mid Y^{t-1}) - h(Y^t_1 \mid Y^{t-1} , x_{1,t}, x_{2,t}) \\
= I(x_{1,t}, x_{2,t}; Y^t_1 \mid Y^{t-1}).
\]

where (a) holds because the state \( x_{i,t}, i = 1, 2 \) is a function of the channel input and output up to the step \( t \). The inequality (b) holds because the conditioning does not increase the entropy.

\[\square\]

[Proof of Lemma 3.3.3] : 

Proof. We first prove the inequality (3.21). Denote \( \mathbf{x}_t = [x_{1,t}, x_{2,t}]^T \), \( \mathbf{u}_t = [u_{1,t}, u_{2,t}]^T \)
and \( \mathbf{w}_t = [w_{1,t}, w_{2,t}]^T \). Define \( A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \) and the recursion of the system can be written in the following vector form:

\[
\mathbf{x}_{t+1} = A\mathbf{x}_t + \mathbf{u}_t + \mathbf{w}_t.
\]  

(3.89)

Following this notation, consider the mutual information

\[
\sum_{t=0}^{T-1} I(x_{1,t}, x_{2,t}; Y_t|Y^{t-1}) = \sum_{t=1}^{T-1} I(x_{1,t}, x_{2,t}; Y_t|Y^{t-1}) + I(x_{1,0}, x_{2,0}; Y_0)
\]

\[
= \sum_{t=1}^{T-1} I(\mathbf{x}_t; Y_t|Y^{t-1}) + I(x_{1,0}, x_{2,0}; Y_0),
\]

where the first term of the right side satisfies

\[
\sum_{t=1}^{T-1} I(\mathbf{x}_t; Y_t|Y^{t-1}) = \sum_{t=1}^{T-1} \left[ h(\mathbf{x}_t|Y^{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
= \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1} + \mathbf{u}_{t-1} + \mathbf{w}_{t-1}|Y^{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
\geq \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1} + \mathbf{u}_{t-1} + \mathbf{w}_{t-1}|Y^{t-1}, \mathbf{u}_{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
= \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1} + \mathbf{w}_{t-1}|Y^{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
\geq \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1} + \mathbf{w}_{t-1}|Y^{t-1}, \mathbf{w}_{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
= \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1}|Y^{t-1}, \mathbf{w}_{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]

\[
= \sum_{t=1}^{T-1} \left[ h(A\mathbf{x}_{t-1}|Y^{t-1}) - h(\mathbf{x}_t|Y^t) \right]
\]
\begin{align*}
\sum_{t=1}^{T-1} \left[ \log |A| + h(x_{t-1}|Y^{t-1}) - h(x_t|Y^t) \right] &
= (T - 1) \log |A| + h(x_0|Y_0) - h(x_{T-1}|Y^{T-1}),
\end{align*}

where the inequality (a) holds because the input $u_{t-1}$ is a function of the channel output sequence $Y^{t-1}$, the inequality (b) holds because conditioning does not increase the entropy, the equality (c) holds because the process noise $w_{t-1}$ is independent of the state at the same time step $x_{t-1}$, and the equality (d) holds because of the property of the differential entropy.

Denote $\lim \sup_{T \to \infty} \mathbb{E} x_i x_i^T = M, i = 1, 2$. Since the system is mean square stable, the covariance matrix $M$ has bounded determinant $|M|$. The property of differential entropy yields that $h(x_{T-1}|Y^{T-1}) \leq \log(2\pi e)^2|M|$.

Thus, the following inequality is obtained,

$$
\sum_{t=1}^{T-1} I(x_t; Y_t|Y^{t-1}) \geq (T - 1) \log |A| + h(x_0|Y_0) - \log 2\pi e |M|. \tag{3.90}
$$

Taking the limit infimum on the average of both the sides of the inequality (3.90), we obtain

$$
\lim \inf_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} I(x_t; Y_t|Y^{t-1}) \geq \lim \inf_{T \to \infty} \frac{1}{T} \left[ (T - 1) \log |A| + h(x_0|Y_0) - \log 2\pi e |M| \right]
\geq \log |A| = \log |\lambda_1| + \log |\lambda_2|.
$$

Following the similar argument, we prove the inequality (3.19) by considering the mutual information.

$$
\sum_{t=0}^{T-1} I(x_{1,t}; Y_t|Y^{t-1}, x_{2,t}) = \sum_{t=1}^{T-1} I(x_{1,t}; Y_t|Y^{t-1}, x_{2,t}) + I(x_{1,0}; Y_0|x_{2,0}). \tag{3.91}
$$

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The first term of the right hand side satisfies

\[
\sum_{t=1}^{T-1} I(x_{1,t}; Y_t|Y^{t-1}, x_{2,t}) = \sum_{t=1}^{T-1} \left[ h(x_{1,t}|Y^{t-1}, x_{2,t}) - h(x_{1,t}|Y^t, x_{2,t}) \right] = \sum_{t=1}^{T-1} \left[ h(\lambda_1 x_{1,t-1} + u_{1,t-1} + w_{1,t-1}|Y^{t-1}, x_{2,t}, u_{1,t-1}) - h(x_{1,t}|Y^t, x_{2,t}) \right] = \sum_{t=1}^{T-1} \left[ h(\lambda_1 x_{1,t-1} + w_{1,t-1}|Y^{t-1}, x_{2,t}, w_{1,t-1}) - h(x_{1,t}|Y^t, x_{2,t}) \right] \geq (a) \sum_{t=1}^{T-1} \left[ h(\lambda_1 x_{1,t-1}|Y^{t-1}, x_{2,t}) - h(x_{1,t}|Y^t, x_{2,t}) \right] = (b) \sum_{t=1}^{T-1} \left[ \log |\lambda_1| + h(x_{1,t-1}|Y^{t-1}, x_{2,t}) - h(x_{1,t}|Y^t, x_{2,t}) \right] \geq (c) \sum_{t=1}^{T-1} \left[ \log |\lambda_1| + h(x_{1,t-1}|Y^{t-1}, x_{2,t}, w_{2,t-1}, u_{2,t-1}) - h(x_{1,t}|Y^t, x_{2,t}) \right] \geq (d) \sum_{t=1}^{T-1} \left[ \log |\lambda_1| + h(x_{1,t-1}|Y^{t-1}, x_{2,t-1}) - h(x_{1,t}|Y^t, x_{2,t}) \right] = (T - 1) \log |\lambda_1| + h(x_{1,0}|Y_0, x_{2,1}) - h(x_{1,T-1}|Y^{T-1}, x_{2,T-1}).
\]

where the inequality \((a)\) holds because conditioning does not increase the entropy, the equality \((b)\) holds because the process noise \(w_{1,t-1}\) is independent of \(Y^{t-1}\) and \(x_{2,t}\), the equality \((c)\) holds because the noise \(w_{2,t-1}\) is independent of \(x_{1,t}\) and the input \(u_{2,t-1}\) is a function of the channel output \(Y^{t-1}\), the equality \((d)\) holds because \(x_{2,t} = \lambda_2 x_{2,t-1} + w_{2,t-1} + u_{2,t-1}\) and \(w_{2,t-1}, u_{2,t-1}\) are known.

The differential entropy \(h(x_{1,T-1}|Y^{T-1}, x_{2,T-1})\) can be bounded in terms of the state variances \(\mathbb{E}x_1^2\). If the system is mean square stable, there exist a positive
number $M_1$ such that $h(x_{1,T-1}|Y^{T-1},x_{2,T-1}) < \log 2\pi e M_1$. Thus, we obtain

$$
\sum_{t=1}^{T-1} I(x_{1,t};Y_t|Y^{t-1},x_{2,t}) \geq (T-1) \log |\lambda_1| + h(x_{1,0}|Y_0, x_{2,1}) - \log 2\pi e M_1. \tag{3.92}
$$

Similarly, by taking the limit infimum to both sides of the inequality (3.92), we have

$$
\liminf_{t \to \infty} \frac{1}{T} \sum_{t=1}^{T-1} I(x_{1,t};Y_t|Y^{t-1},x_{2,t}) \geq \log |\lambda_1|. \tag{3.93}
$$

By symmetry, we can also prove the second inequality (3.20).

3.6.3 Existence of Solution for Correlation Coefficient

In this section of Appendix, we provide proofs for the existence of solutions of the nonlinear equations associated with the coefficients that appear in Theorem 3.2.5 and Corollary 1.

**Proposition 3.6.3.** Given that $\lambda_1 \lambda_2 < 0$, the nonlinear equation (3.42) has at least one solution.

**Proof.** By multiplying the term $P_1 + P_2 + 2|\rho|\sqrt{P_1 P_2} + N$ to both sides of the nonlinear equation (3.42), we obtain

$$
\rho(P_1 + P_2 + 2|\rho|\sqrt{P_1 P_2} + N) = \lambda_1 \lambda_2 (N\rho - \text{sgn}\sqrt{P_1 P_2}(1 - \rho^2))
$$

$$
+ \gamma \sqrt{(P_1 + P_2 + 2|\rho|\sqrt{P_1 P_2} + N - \lambda_1^2(N + P_2(1 - \rho^2))}
$$

$$
\cdot \sqrt{(P_1 + P_2 + 2|\rho|\sqrt{P_1 P_2} + N - \lambda_2^2(N + P_1(1 - \rho^2)))}.
$$

Let $\rho = 0$, the left hand side is equal to 0, while the right hand side is equal to

$$
\lambda_1 \lambda_2(-\sqrt{P_1 P_2}) + \gamma \sqrt{(P_1 + P_2 + N - \lambda_1^2(N + P_2))\sqrt{(P_1 + P_2 + N - \lambda_2^2(N + P_1))}}. \tag{3.94}
$$

66
By $\lambda_1 \lambda_2 < 0$, the right hand side is greater than zero, i.e. the right hand side is greater than the left hand side when $\rho = 0$. When $\rho = 1$, the right hand side of (3.42) is written as

$$P_1 + P_2 + \sqrt{P_1 P_2} + N > \lambda_1 \lambda_2 N + \gamma \sqrt{P_1 + P_2 + 2 \sqrt{P_1 P_2} + N - \lambda_2^2 N} \cdot \sqrt{P_1 + P_2 + 2 \sqrt{P_1 P_2} + N - \lambda_1^2 N}.$$

By the fact that $\gamma < 1$, $\lambda_1 \lambda_2 < 0$, we can see that the right hand side is less than the left hand side,

Thus, there must exist at least a solution to the nonlinear equation with the value within the interval $(0, 1)$.

Remark 3.6.4. We can see that when $\lambda_1 \lambda_2 > 0$, the similar argument in the proof of Proposition 3.6.3 cannot be established to prove the existence of solution to equation (3.42).

Proposition 3.6.5. The nonlinear equation (3.58), which is equivalent to (3.42) with $\gamma = 0$, has at least one solution for $|\rho|$.

Proof. Recall that equation (3.58) is written as follows.

$$\rho^2 = \left( \frac{\lambda_1 \lambda_2 (N \rho - \text{sgn}(\rho) \sqrt{P_1 P_2 (1 - \rho^2)})}{P_1 + P_2 + 2|\rho| \sqrt{P_1 P_2} + N} \right)^2. \quad (3.95)$$

When $\rho = 0$, the left hand side is equal to zero, and the right hand side is

$$\frac{P_1 P_2}{(P_1 + P_2 + N)^2} > 0 \quad (3.96)$$

When $\rho = 1$, the left hand side is equal to 1, and the right hand side is

$$\frac{N^2}{(P_1 + P_2 + \sqrt{P_1 P_2} + N)} \quad (3.97)$$
According to the necessary condition (3.4), we know that the sum rate condition must be satisfied,

$$\log |\lambda_1| + \log |\lambda_2| \leq \frac{1}{2} \log \left( 1 + \frac{P_1 + P_2 + 2\rho \sqrt{P_1 P_2}}{N} \right) \quad (3.98)$$

Therefore, the right hand side (3.97) must be less than 1.

In summary, the right hand side is greater than the left hand side when $\rho = 0$ and then the right hand side is less than the left hand side when $\rho = 1$. Thus, there must exist a solution to the nonlinear equation with the abstract value $0 < |\rho| < 1$. \qed
CHAPTER 4

PRICING POLICIES FOR PLUG-IN ELECTRIC VEHICLE CHARGING STATIONS

4.1 Introduction

Electric vehicles are recognized as the future resolution for reducing the air pollution from consuming fossil fuels. Among all the types of electric vehicles, the plug-in electric vehicle (PEV), which relies solely on electrical power provided by an on-board battery, without consuming any fossil fuel, is expected to be the main stream vehicle in the next generation. The penetration level of the PEVs in the future requires a new framework of infrastructure in the metropolitan area. Public or commercial PEV charging infrastructure must be constructed in an efficient way to promote the adoption of PEVs.

In this chapter, we are particularly interested in analyzing commercial PEV charging stations. The charging station business requires comprehensive study of the location, queueing effect, waiting time and pricing of the service station, competition of stations in other locations, etc. We will conduct a combined analysis of the pricing model for the commercial station with consideration of various aspects.

In recent years, there have been studies in the business model of the firms that provide services for customers waiting in a queue. In this setting, the customer select firms not only based on the service price, but also on the waiting cost. To design the service price, firms must be aware of the characteristics of the queueing effect. Such problems have been considered in many existing works. The queueing effect in
the service price is first studied in [57] where a threshold of queue length is given as a critical to maximize the welfare of all customers. A toll is imposed to maintain a queue in the optimal average length. The real-time pricing strategy that adapts to the changing queue length is considered in [10]. The price and market share of a monopoly make-to-order firm is considered in [9] where the price of the firm and service rate can be chosen according to the number of customers joining the firm in order to maximize the firm revenue. The traditional relationship of price and demand is shown to be greatly changed. Besides the single firm setting, competition of more than one firms with queueing customers are considered in [11, 40].

Besides the waiting cost, the traveling cost is also an important factor in customer’s decision. In reality, the customers may come from one of discrete locations in a geographical area. A closer station will be more attractive for customers at the same location in the sense of travel cost. Especially for electric vehicle owners, the battery may only sustain for limited miles and the nearest charging station will be the primary choice. The location planning of service facilities are widely studied in transportation and operational research. The well-known p-median or p-center problem [37, 58] focus on finding a set of optimal locations among candidate sites. But those problems focus on minimizing the average weighted travel cost only. Moreover, p-median or p-center problem are centralized optimization problems based on the assumption that a social planner makes the placement of each facilities. Competition between selfish agents who select locations are studied in [8, 13]. However, the pricing strategy is not discussed.

In this chapter, both waiting and traveling cost of the customer are considered in the price competition of two service stations. In our setting, the customers are present at various locations in a region. They need to select one station among all possible choices. The cost that the customer sees when it goes to a particular station is the cost to drive to the station, the waiting cost at the station and the price the
station charges. The stations need to decide what prices to charge to maximize their revenues. A higher price may lead to a lower revenue by driving away customers, while a lower price also lead to a lower revenue by increasing the waiting cost. For instance, the customers may represent equipment (e.g. vehicles) that needs to be serviced at one of many maintenance shops (e.g. garages). The waiting cost represents time spent at the shop due to other customers being present and due to servicing time of ones own equipment. By setting a low price, a shop may be able to attract customers from a location far away. Another example is the plug-in electric vehicle (PEV) charging business. The customers may represent PEVs that need to be charged at one of many commercial charging stations that offer level 3 charging (and hence quick charging). The distance dependent cost term represents the electricity and time cost to derive to a further station.

We focus particularly on the price competition of the service stations, particularly the case that the total demand is scarce and both stations need to secure enough market share by setting prices that are lower than the usual price in the monopoly scenario. We adopt a game-theoretic framework to study this price competition. Both are modeled as players in the pricing game, with their revenues as payoffs. The strategy of each player is one’s price, which is positive and continuous-valued that is similar to the prices in Cournot or Betrand games [15]. We are interested in the price profile as a combination of all prices and to see if a price profile is a Nash equilibrium of this pricing game. In the Nash equilibrium, each player’s strategy is the best response to the other player’s strategy. Neither of players can be better off by unilaterally deviating from the price in the Nash equilibrium.

In this chapter, we adopt a two-level hierarchical game formulation to study this problem. The hierarchical game is a combination of games in multilevels and the higher level game is played based in consideration of the equilibrium of the next lower level game [? ? ? ? ]. Specifically, in our problem, the customers can be modeled
as players of a congestion game in the lower level. In this level, the customers play against each other to choose the best station to charge based on the travel cost and congestion level. This fits very well into the framework of the congestion game or selfish routing game. On the high level, the stations are players in a pricing game, who chooses price to compete for higher revenue. For each combination of the strategies, there is a corresponding Nash equilibrium formed by the lower level game. The decision making of the service station must take the outcome of the choices of the customers as the players in the lower level game.

We formulate nonlinear programs that maximize the revenues of the stations to find the Nash equilibrium. Each nonlinear program is to find the best response for each station given other station’s strategy. The constraints of the nonlinear programs reflects the customer’s response to the price profiles. However, the constraint is difficult to characterize because there is no explicit form to describe the relationship between each price profile and the customer assignment. In recognition of the difficulty in determining the feasible set, we first solve the nonlinear program within a subset of price profile and customer assignment, which corresponds to a particular scenario. For instance, for a two station and two customer locations setting, one station may attract customers from one node and the other attract customers from another node. The feasible set can be easily characterized in this setting and the solution as a price pair can be obtained. The main result is the sufficient condition for such a solution in a particular scenario being a Nash equilibrium of the pricing game. The condition can be used as a guideline to search for a Nash equilibrium. For the presentation ease, we focus on the case with two stations and two customer locations. The result can be extended to the case with a general number of customers nodes.

This chapter is organized as follows. Section 4.2 presents the model of locations of customer and stations, customer queueing model and decision rules. The pricing
game of two stations are also formulated. Section 4.3 presents the main results for two scenarios of customer and station location layouts. Section 4.4 presents the proofs of the main results. Section 5.6 concludes the chapter.

4.2 Problem Setting and Assumptions

4.2.1 PEV Charging Station and Customer Model

We model a geographical area as being represented by a connected, weighted and undirected graph with node set $\mathcal{N} = \{1, 2, \ldots, N\}$. The nodes represent centers of demand generation (for instance, parking lots, university or office campuses, malls or suburbs). On a given subset of these $n$ nodes, a charging station is located. We assume $m$ stations in total. An edge between two adjacent nodes denotes the road connecting the two nodes, with the weight being the driving distance. Denote the length of the shortest path connecting them by $d_{ij}$ for two nodes $i$ and $j$.

For simplicity, we model the demand for power at any node $i$ to be generated according to a time homogeneous Poisson process with intensity $\Lambda_i$. A customer generated at node $i$ selects a charging station located at $j$ to minimize her cost function as discussed later. A charging station located at node $j$ is modeled as an $M/M/1$ queue, with a service rate $\mu$. For simplicity, we assume that $\mu_j = \mu, \forall j$ in this note; so that the average service time for a customer is independent of the charging station location. Moreover, the charging station levies price $p_j$ to charge a PEV.

We now describe the decision rule followed by the customers. The cost suggested by a customer has these components: the distance driving to a charging station, the price paid and the waiting cost. We model the waiting cost to be proportional to the average waiting time in the queue. Denote the utility that a customer received by
choosing the \( j \)-th node to charge as \( U_j \). The customer’s decision rule is,

\[
\arg \min_{j \in \mathcal{N}_m} U_j = \begin{cases} 
  v & j = 0, \\
  R - d_j - p_j - \frac{h}{\mu - \lambda_j} & j \in \mathcal{N},
\end{cases}
\tag{4.1}
\]

where the different terms are described as follows.

- The minimization is done over the set \( \mathcal{N}_m = 0 \cup \mathcal{N} \) of locations of charging stations and the node 0 which is defined as some alternative such as a decision to not charge or to charge at a residence, the utility associated with that alternative choice is set to be \( v > 0 \).

- \( R \) denotes the benefit that a customer received by charging the PEV immediately. It is constant for all nodes \( j \).

- \( d_j \) denotes the distance driven to charging station at node \( j \). It may include a proportionality constant to convert distance to cost.

- \( p_j \) denotes the price paid to charging station \( j \).

- \( \frac{h}{\mu - \lambda_j} \) models the waiting cost for station at node \( j \). It is derived from the \( M/M/1 \) model of queue servicing with \( h \) being a constant of proportionality and \( \lambda_j \) being the rate at which customers join the queue at station \( j \). To make the queue stable, we assume \( \mu - \lambda_j > 0 \).

By convention, we use the term \( R_{ij} = R - d_j^l \) for the valuation of the customer at node \( i \) choosing station \( j \).

**Assumption 7.** The value \( R_{ij} \) is assumed to be large enough so that the customer chooses some station when there is no customer in the queue and the service price is zero, i.e. we can assume that \( R_{ij} > \frac{h}{\mu} \).

Given this decision rule for the customers, each station \( j \)’s problem is to select the price \( \{p_j\} \) to maximize the revenue computed as \( \pi_j = p_j \lambda_j \).

### 4.2.2 Hierarchical Game Formulation: Lower Level Congestion Game

To study the pricing strategies of the charging stations and behaviors of the PEV customers, a hierarchical game perspective is adopted. Suppose the prices
of the charging stations $p_j, j \in \mathcal{N}$ are fixed. Then the customers will make their decisions based on the rules described in (4.1). Suppose customers are able to observe the choices of the other customers and each customer is a selfish agent who only maximizes his utility. Thus, the decision process of the customers can be modeled as selfish routing games or congestion games. We assume that the result of selfish routing reaches the Nash equilibrium of the congestion game after a long time of evolvement.

On top of this congestion game, the price competition between charging stations can be modeled as a pricing game. The action of the station, which is the price, will affect the customers distribution among the stations, which is corresponding to the Nash equilibrium of the congestion game in the lower level. Thus, the price is set by each station to maximize the station’s revenue with consideration of the Nash equilibrium of the congestion game in the lower level.

Formally, we first define the congestion game in the lower level as follows. The demand generation rate $\Lambda_i$ at the customer node $i$ can be viewed as the nonatomic measure of the customer since this demand rate will be distributed to the nodes in the set $\mathcal{N}_m$ and the value is continuous. This was used in to model a game with a large number of players in which the individual effect of the player can be neglected. The only significant impact will be caused by a coalition of the players.

In this perspective, the lower level game can be viewed as a nonatomic game. The action set of the nonatomic player is $\mathcal{N}_m = \{0, 1, 2, \ldots, N\}$, which represents the choice of a particular charging station or the alternative. The utility function of the nonatomic player from node $i$ with the action $j$ can be written as Denote the utility of the customer at node $i$ who takes action $j$ as

$$u_{ij} = \begin{cases} v & \text{if } j = 0, \\ R_{ij} - p_j - \frac{h}{\mu - \lambda_j} & \text{otherwise.} \end{cases}$$

(4.2)
where $R_{ij} = R - d_{ij}$ is the utility related to the travel distance, which is an individual utility. The term $\frac{b_i}{\mu - \lambda_j}$ is the congestion level of the station $j$.

This game is shown to have at least one pure-strategy Nash equilibrium. At the Nash equilibrium, each player’s action $\sigma(i) = \arg\max_j u_{ij}$.

For each price profile, the customers at each node play a nonatomic congestion game at the lower level. We denote the lower level game as $G_l(p_1, p_2)$. For customers at node $i = a, b$ who takes the action $j$. To model the actual customer assignment at the equilibrium, define a state vector $x = [\lambda_{a0}, \lambda_{a1}, \lambda_{a2}, \lambda_{b0}, \lambda_{b1}, \lambda_{b2}]^T$, where $\lambda_{ij}$ denotes the rate of the customers from the node $i$ who takes action $j \in \{0, 1, 2\}$, where $j = 1, 2$ means the customer choose to charge at station $j$ and $j = 0$ stands for alternative choice of charging elsewhere instead of going to any station.

**Proposition 4.2.1.** Denote the binary variable $y_{ij} = 1 \{\lambda_{ij} > 0\}$ to represent that there are positive number of customers at node $i$ who takes action $j$. At the Nash equilibrium of the lower level congestion game, for each node $i$, the following inequalities hold.

\[
y_{ij_1}y_{ij_2}(u_{ij_1} - u_{ij_2}) = 0, \quad (4.3)
\]
\[
y_{ij_1}(1 - y_{ij_2})(u_{ij_1} - u_{ij_2}) \geq 0. \quad (4.4)
\]

where $u_{ij}$ is utility of the customer taking the action $j$, which is given in (4.2).

**Proof.** The equality (4.3) holds because if $\lambda_{ij_1} > 0$ and $\lambda_{ij_2} > 0$, then the utility of the two actions $u_{ij_1} = u_{ij_2}$, otherwise, the customers will achieve higher utility by deviating from the current action, which is contradictory to the definition of Nash equilibrium. The inequality (4.4) holds because if no customers choose action $j_2$, then the utility of action $j_1$ must be no less than the action $j_2$ in order to be consistent with Nash equilibrium. \qed
Proposition 4.2.1 can be used as a guideline to determine whether the vector $x$ is associated with a Nash equilibrium of the lower level game $G_l(p_1, p_2)$.

4.2.3 Hierarchical Game Formulation: High Level Pricing Game

The upper level game is defined between the stations. For each station, the strategy set is the continuous positive real number $p_j > 0$, which is the price. The utility of the station is the revenue defined as $u_j = p_j \lambda_j$. The set of prices of all stations $P = \{p_1, p_2, \ldots, p_n\}$ is the strategy profile.

Note that for each price profile $P$, there is a related congestion game for the customers, which determines the joining rate $\lambda_j, j \in N$. The pricing strategy depends on the interaction between the price and the result of the lower level congestion game. A strategy profile is a Nash equilibrium of the game between the stations if and only if

$$u_j(p_j^*, p_{-j}^*) \geq u_j(p_j, p_{-j}^*), p_j > 0$$

(4.5)

for all $j \in N$ and $p_{-j} = \{p_1, p_2, \ldots, p_{j-1}, p_{j+1}, \ldots, p_n\}$.

It is desirable that the strategy profile of the charging stations to achieve the Nash equilibrium. Given the model of hierarchical game, we want to know the condition that the Nash equilibrium of the station’s pricing game exists and how to characterize it. However, since the Nash equilibrium of the lower level game is difficult to be obtained analytically, we adopt a heuristic approach to study this problem.

As an initial attempt on the pricing game, we first consider a specific case with two station nodes and two customer nodes, i.e. $m = 2, N = 2$. Without loss of generality, each station node does not have any customer. Two stations are denoted as station 1 and 2. The two customers nodes are denoted as node $a$ and $b$.

In the upper level, the two stations 1, 2 are modeled as selfish, noncooperative
players. The payoff of each player is defined as the revenue \( \pi_i, i = 1, 2 \). The strategy for each player is the price of each station \( p_i \). Each station \( i \) maximizes the payoff by choosing the strategy \( p_i \) in response to the other’s strategy. Thus, the payoff \( \pi_i(p_1, p_2), i = 1, 2 \) is a function of the pair of strategies or prices denoted as the strategy or price profile \( (p_1, p_2) \). We denote this game as \( G_{ab} \). The similar setting of pricing game in the scenario of two competitors with customers in one location is discussed in [11].

Define the strategy profile at the Nash equilibrium as \( (p_1^*, p_2^*) \), such that

\[
\begin{align*}
\pi_1(p_1^*, p_2^*) &= \max_{p_1} \pi_1(p_1, p_2^*), p_1 > 0, \\
\pi_2(p_1^*, p_2^*) &= \max_{p_2} \pi_2(p_1^*, p_2), p_2 > 0.
\end{align*}
\]

In this chapter, we are interested in the Nash equilibrium of the game \( G_{ab} \) which is the solution of the following two nonlinear programs.

\[
P_1 : \max_{p_1, \lambda_1} \pi_1 = p_1 \lambda_1 \tag{4.8}
\]

\[
s.t. \quad p_1 > 0, \lambda_1 = \lambda_{a1} + \lambda_{b1}, \\
(p_1, p_2, x) \in F. \tag{4.9}
\]

\[
P_2 : \max_{p_2, \lambda_2} \pi_2 = p_2 \lambda_2 \tag{4.10}
\]

\[
s.t. \quad p_2 > 0, \lambda_2 = \lambda_{a2} + \lambda_{b2}, \\
(p_1, p_2, x) \in F. \tag{4.11}
\]

where \( F \) is the feasible set of the price profiles and the state vector, which satisfies the condition (4.3) and (4.4).

In order to solve the problems \( P_1, P_2 \), the feasible set \( F \) needs to be specified. However, in different layouts of customer nodes and station nodes, the states in
TABLE 4.1
ALL POSSIBLE NODE LAYOUTS

<table>
<thead>
<tr>
<th></th>
<th>$R_{a1} &gt; R_{b1}$</th>
<th>$R_{a1} &lt; R_{b1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{a2} &gt; R_{b2}$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$R_{a2} &lt; R_{b2}$</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

response to the price profile will be different. Located in a two dimensional space, there are infinitely possible layouts. Fortunately, we can classify the layouts based on distances of the two customer nodes to the stations which are reflected in $R_{i,j}$ values. For instance, the node $a$ may be closer to station 1 than station 2, and also node $b$ is closer to station 2 than station 1. This relationship can be written as $R_{a1} > R_{a2}$, $R_{b1} < R_{b2}$. Table 4.1 lists all possible layouts regarding to the distances between the nodes and stations.

We say that both stations share a customer node when the two stations have market share at the same node. This implies that $u_{i1} = u_{i2} \geq v$.

**Proposition 4.2.2.** If $R_{i1} - R_{i2} \neq R_{j1} - R_{j2}$, then both stations 1 and 2 cannot share more than one single customer node for each price profile.

**Proof.** We prove by contradiction. Suppose both stations form equilibria at both nodes $i$ and $j$. The customer joining rate of station 1 and 2 are $\lambda_1$, $\lambda_2$ respectively. Then, the following equalities must hold,

$$R_{i1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{i2} - p_2 - \frac{h}{\mu - \lambda_2}, \quad (4.12)$$
$$R_{j1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{j2} - p_2 - \frac{h}{\mu - \lambda_2}. \quad (4.13)$$

Since $R_{i1} - R_{j1} \neq R_{i2} - R_{j2}$, the two equalities cannot hold simultaneously. \qed
Without loss of generality, we make the following assumption.

**Assumption 8.** \( R_{a1} - R_{a2} > R_{b1} - R_{b2} \).

Assumption 8 is made to focus on the case that two station only share one customer node. For the case that \( R_{a1} - R_{a2} < R_{b1} - R_{b2} \), we can use symmetry to derive similar results which will be presented later. Since Assumption 8 implies \( R_{a1} - R_{b1} > R_{a2} - R_{b2} \), the case 2 never happens.

Note that if there are excessive demand at the customer nodes, i.e., the value of \( \Lambda_a \) and \( \Lambda_b \) are very large, then, based on discussions in [11], it is possible that two stations reach the optimal revenue by only attracting a part of the customers, without considering the other station’s strategy. There are still customer not being attracted in this situation. There will be no competition between the two stations. To exclude this case, we make the following assumption.

**Assumption 9.** Each of the demand generating rates \( \Lambda_a \) and \( \Lambda_b \) is insufficient in the sense that neither station can achieve optimal revenue by only attracting customers from a single node.

The situation described in Assumption 9 is compliant with the highly competitive case described in [11, Theorem 6]. This assumption implies that both stations set prices low enough to compete for customers such that there is customer left unattracted. Thus, the equality \( \lambda_1 + \lambda_2 = \Lambda_a + \Lambda_b \) holds. The Assumption 9 usually implies that the demand is scarce and two stations need to compete for customers at a certain customer node, i.e., two stations share a customer node. If stations share a node and the layouts of the nodes are given, then it is easy to characterize the relations between the price profiles and the states. Thus, we can solve the nonlinear programs (4.8) and (4.10) within a specified feasible set. The solution can be further studied to see if it is a Nash equilibrium. Sufficient conditions for the solutions obtained in each market equilibrium in each scenarios will be presented in the main
result. The scenarios with an equilibrium formed at node $a$ in both case 3 and 4 are studied in section 4.3.1 by symmetry. The scenario with an equilibrium formed at node $b$ in the case 1 is covered in section 4.3.2. The discussion in this section can also be extended to the scenario with an equilibrium formed at node $b$ in the case 4 by symmetry. The table 4.2 summarizes the different cases that will be discussed in the following sections.

4.3 Main Results

In this section, we present the main results obtained for each scenario.
4.3.1 The Equilibrium at Node $a$ with $R_{a2} < R_{b2}$

We first focus on the scenario that both stations share the node $a$ in both cases $\circled{3}$ and $\circled{4}$. In other word, the following condition holds.

$$R_{a2} < R_{b2}. \quad (4.14)$$

By Assumption 9, both station will set prices to compete for customers from both nodes, then it is impossible to have the case that $\lambda_{a0} > 0$ or $\lambda_{b0} > 0$ at the Nash equilibrium of the congestion game. Moreover, since both stations will enter a highly competitive situation, then it is very possible that both station share a particular node. Thus, we first consider the case that both station share a node and study the Nash equilibrium of the pricing game in that case.

**Lemma 4.3.1.** If the condition $\square (4.14)$ holds and both stations share the node $a$, i.e.
the inequality holds

\[ R_{a2} - p_2 - \frac{h}{\mu - \lambda_2} = R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} \geq v. \]  

(4.15)

where \( \lambda_i \) denotes the total density of customers choosing station \( i \), then we have

\[ \lambda_1 = \Lambda_a - \lambda, \]  

(4.16)

\[ \lambda_2 = \lambda + \Lambda_b, \]  

(4.17)

where \( 0 \leq \lambda < \Lambda_a \) means the portion of customers at node \( a \) joining station \( 2 \).

Proof. By (4.15), we know that both stations provide equal valuation to the customers at node \( a \), and also notice that this valuation is strictly greater than the alternative, so all customers at node \( a \) are choosing to charge at either station. The density of customers choosing station \( 1 \) is \( \lambda_1 = \Lambda_a - \lambda \), where \( \lambda > 0 \). By (4.15) and Assumption
In the following inequality can be obtained.

\[ R_{b2} - p_2 - \frac{h}{\mu - \lambda_2} > R_{b1} - p_1 - \frac{h}{\mu - \lambda_1}. \] (4.18)

This indicates that at node \( b \), the valuation of station 1 is strictly less than the one provided by station 2, thus, no customers at node \( b \) chooses station 1. In other words, the station 1 can only attract part of customers at node \( a \), thus (4.16) holds. By (4.14) and (4.15), we obtain the following inequality,

\[ R_{b2} - p_2 - \frac{h}{\mu - \lambda_2} > v. \] (4.19)

This indicates that for the customers at node \( b \), the station 2 provides a valuation with a value strictly greater than the alternative.

By (4.18) and (4.19), we can prove that all the customers at node \( b \) will be
attracted to station 2 using contradiction. Suppose there are some customers at node $b$ who do not choose station 2. By the inequality (4.18), they cannot have higher valuation by choosing station 1. Also due to (4.19), they still have a higher valuation in choosing station 2 than the alternative value $v$. So these customers will still choose station 2, which leads to a contradiction.

With $\lambda$ as the portion of customers at node $a$ attracted to station 2, we can prove that $\Lambda_b = \lambda + \Lambda_a$.

Lemma 4.3.1 indicates that due to the station 2’s relative advantage at node $b$, if two stations share node $a$, then station 2 should be able to attract all customers at node $b$. From Lemma 4.3.1 we can see that station 1 has more advantage at node $a$ over station 2, and station 2 is advantageous at node $b$. Thus, once the two stations share node $a$, the lower level game reaches an Nash equilibrium that can be described
in the set $\mathcal{X}_a = \{[0, \Lambda_a - \lambda, \lambda, 0, 0, \Lambda_b]^{T} | 0 \leq \lambda < \Lambda_a \}$. The feasible set for $(p_1, p_2, x)$ is defined as

$$F_a = \{(p_1, p_2, x) \mid R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{a2} - p_2$$

$$- \frac{h}{\mu - \lambda_2} \geq v, \lambda_1 = \lambda_{a1} + \lambda_{b1}, \lambda_2 = \lambda_{a2} + \lambda_{b2}, x \in \mathcal{X}_a \}.$$  

We illustrate one possible scenario in the state set $F_a$ of the layout case 3 in Figure 4.5. We use pie charts to represent the node $a$ and $b$. The dark blue area represents the market share of the station 1 and red area represents the market share of the station 2.

Clearly, $F_a \subset F$. We consider the game of the two stations defined by (4.8) and (4.10) over the subset $F_a$. Denote this game as $G_a$.

Using the condition in Lemma 4.3.1, the Nash equilibrium of game $G_a$ can be
found by jointly solving the following nonlinear programs.

$$\max_{p_1, \lambda_2} \pi_1 = p_1 \lambda_1$$  \hspace{1cm} (4.20)

s.t. \hspace{0.5cm} R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} \geq v, \\
\hspace{1cm} R_{a2} - p_2 = R_{a1} - p_1 - \frac{h}{\mu - \lambda_1}, \\
\hspace{1cm} \lambda_1 + \lambda_2 = \Lambda_a + \Lambda_b, \\
\hspace{1cm} p_1 > 0, 0 < \lambda_1 \leq \Lambda_a, \hspace{0.5cm} \lambda_1 < \mu.

$$\max_{p_2, \lambda_2} \pi_2 = p_2 \lambda_2$$  \hspace{1cm} (4.21)

s.t. \hspace{0.5cm} R_{a2} - p_2 - \frac{h}{\mu - \lambda_2} \geq v, \\
\hspace{1cm} \lambda_1 + \lambda_2 = \Lambda_a + \Lambda_b, \\
\hspace{1cm} p_2 > 0, \Lambda_b \leq \lambda_2 \leq \Lambda_a + \Lambda_b, \hspace{0.5cm} \lambda_2 < \mu.

The following constraints are satisfied by both programs

$$R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_2},$$

$$\lambda_1 + \lambda_2 = \Lambda_a + \Lambda_b.$$  \hspace{1cm} (4.22)

where the price profile \((p_{1,a}^*, p_{2,a}^*)\) denotes the Nash equilibrium of the game \(G_a\). The two nonlinear programs \((4.20)\) and \((4.21)\) computes the best response for station 2 and station 1 over the feasible set \(\in F_a\). Since the solution \(p_i\) of each problem depends on the solution of the other problem \(p_j\), both problems should be solved jointly to obtain the pair of best responses to for each other. The solution of \((4.20)\) and \((4.21)\) can be found by formulating the two nonlinear programs as a nonlinear complementary problem (NCP), using the Karush Kuhn Tucker (KKT) conditions of the two nonlinear programs.
The pair \((p_1, p_2)\) is a solution to nonlinear programs (4.20) and (4.21) only if the solution satisfies the following NCP.

\[
\frac{\partial L_i}{\partial p_1} \geq 0 \perp p_1 \geq 0, \\
\frac{\partial L_i}{\partial \lambda_1} \geq 0 \perp p_1 \geq 0, \\
R_{ai} - p_i - \frac{h}{\mu - \lambda_i} - v \geq 0 \perp \eta_{1,i} \geq 0, \\
R_{ai} - p_i - \frac{h}{\mu - \lambda_i} - \left( R_{aj} - p_j - \frac{h}{\mu - (\Lambda_a + \Lambda_b - \lambda_j)} \right) \perp \eta_{2,i} \geq 0,
\]

for \(i = 1, 2\). Here, \(L_i\) is the Lagrangian of the nonlinear program to compute the best response of station \(i\), defined as

\[
L_i = p_i \lambda_i + \eta_{1,i} \left( R_{ia} - p_i - \frac{h}{\mu - \lambda_i} - v \right) + \eta_{2,i} \left( R_{ai} - p_i - \frac{h}{\mu - \lambda_i} - \left( R_{aj} - p_j - \frac{h}{\mu - (\Lambda_a + \Lambda_b - \lambda_j)} \right) \right) = 0.
\]

The NCP problem comes from the Karush Kuhn Tucker (KKT) conditions of the two nonlinear programs. Since the revenue function is not convex in the decision variables, inequalities are actually necessary conditions for \((p_{c1,a}, p_{c2,a})\) being a solution to nonlinear programs (4.20) and (4.21).

Given that the solution \((p_{c1,a}, p_{c2,a})\) exits, it may not necessarily be the Nash equilibrium of the game \(G_{ab}\). This is because the game \(G_a\) is defined with a smaller feasible set for price profiles. This implies that there may exists an alternative price for one of the stations that achieves higher revenues outside the set \(F_a\). For instance, it is possible that station 2 changes the price to form an equilibrium at node \(b\) to achieve higher revenue than \(p_{c2,a}\). The following theorem provides a sufficient condition to verify that Nash equilibrium \((p_{c1,a}, p_{c2,a})\) of game \(G_a\) is the Nash equilibrium of the game \(G_{ab}\).
Theorem 4.3.2. Suppose the game \( G_a \) has a Nash equilibrium \((p^c_{1,a}, p^c_{2,a})\).

- The strategy profile \((p^c_{1,a}, p^c_{2,a})\) is a Nash equilibrium of the game \( G_{ab} \) if

\[
R_{b1} - p^c_{1,a} - \frac{h}{\mu - \Lambda_a} \leq v, \tag{4.23}
\]

- Given that the condition (4.23) does not hold, the strategy profile \((p^c_{1,a}, p^c_{2,a})\) is a Nash equilibrium of the game \( G_{ab} \) if

\[
\pi_2(p^c_{1,a}, p^c_{2,a}) \geq \pi_2(p^c_{1,a}, p^r_{2,b}) \tag{4.24}
\]

where \( p^r_{2,b} \) is the solution to the following nonlinear program.

\[
\begin{align*}
\max_{p_{2,b}} & \pi_2(p^c_{1,a}, p_{2,b}) = (\Lambda_b - \lambda)p_{2,b} \\
\text{s.t.} & \quad R_{b1} - p^c_{1,a} - \frac{h}{\mu - (\Lambda_a + \lambda)} = R_{b2} - p_{2,b} - \frac{h}{\mu - (\Lambda_b - \lambda)}, \\
& \quad p_{2,b} > 0, \quad 0 \leq \lambda \leq \Lambda_b, \quad \Lambda_a + \lambda < \mu.
\end{align*}
\] (4.25) (4.26)

Remark 4.3.3. The condition (4.23) implies that station 1 can never attract the customers at node \( b \) with the price fixed at \( p^c_{1,a} \) even without the competition of the station 2. If that happens, then it can be proved that \((p^c_{1,a}, p^c_{2,a})\) is Nash equilibrium of the game \( G_{ab} \).

When condition (4.23) is violated, it is possible that station 2 sets a high enough price such that the part of the customers at node \( b \) switch to station 1 with the price \( p^c_{1,a} \). In this case, the state \( x \) at Nash equilibrium of the lower level game will change to the set \( \mathcal{X}_b = \{[0, \Lambda_a, 0, 0, \lambda, \Lambda_b - \lambda]^T, 0 \leq \lambda < \Lambda_b \} \) which represents the set of states in which station 1 attracts customers from node \( b \) and share node \( b \) with station 2.

The price \( p^r_{2,b} \) represents an alternative price of station 2 that achieve the optimal utility within the set \( \mathcal{X}_b \), i.e. the solution to the nonlinear program (4.52). In other words, when the price of station 1 is fixed at \( p^c_{1,a} \), the station 2 can increase its price to the value \( p^r_{2,b} \) to see if the station 2 can get a better utility than the utility achieved by the price \( p^c_{2,a} \). The sufficient condition indicates that if such an alternative cannot
bring a higher utility than \( p_{2,a}^c \), then the best response for station 2 is \( p_{2,a}^c \).

4.3.2 The Equilibrium at Node \( a \) in the Case (1)

According to the definition of the case (1), the following inequality holds.

\[
R_{b1} < R_{a1}, \  R_{b2} < R_{a2}.
\] (4.27)

Similar to the section 4.3.1, we first focus on the game of the two stations under the constraint that a market equilibrium is formed at the node \( a \). The prices of the stations in this scenario must satisfy

\[
R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_2} \geq v. \tag{4.28}
\]

Recall that Lemma 4.3.1 in the section 4.3.1 states that in the scenario (3), once both stations share node \( a \), the entire customers at node \( b \) will join the station 2 and the state set can be specified. In the scenario (1), however, such a property no longer holds. By (4.27), we never know for sure that the node \( b \) will be covered by station 1 or 2.

We use \( G_{a1} \) to denote the game in this scenario. Since, the actual customer joining rate at node \( b \) is unknown, Different values in \( \lambda_{b2} \), the rate of customers joining station 2 at node \( b \), lead to different constraints in the nonlinear programs to obtain the Nash equilibrium of the game \( G_{a1} \). We will discuss the Nash equilibrium in the game \( G_{a1} \) and its connection to the game \( G_{ab} \).

The first case is that the price of station 2 cannot attract any customers from
node \( b \), which leads to the following feasible set

\[
F_{1,a} = \left\{(p_1, p_2, x) \left| R_{b2} - p_2 - \frac{h}{\mu - \lambda_2} - v \leq 0, \right. \right. \\
R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_2} \geq v, \right. \\
\left. \lambda_2 = \lambda_{a2}, \lambda_{b2} = 0, \lambda_1 = \lambda_{a1} \right\}
\] (4.29)

Another case is that the positive portion of customers from node \( b \) is attracted. The related feasible set is

\[
F_{2,a} = \left\{ R_{b2} - p_2 - \frac{h}{\mu - \lambda_2} - v \geq 0, \left. \right. \\
R_{a1} - p_1 - \frac{h}{\mu - \lambda_1} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_2} \geq v, \right. \\
\lambda_2 = \lambda_{a2} + \lambda_{b2}, 0 < \lambda_{b2} \leq \Lambda_b, \lambda_1 = \lambda_{a1} \right\}.
\] (4.30)

Remark 4.3.4. In this setting, station 1 cannot attract customers from node \( b \), i.e. \( \lambda_{b1} = 0 \). If \( \lambda_{b1} > 0 \), then either station 1 will attract all customers from both nodes or form an equilibrium at node \( b \) instead of \( a \). This can be shown by using Assumption

The pricing game \( G_{a1} \) is the game with feasible set \( F_{a1} \) and \( F_{a2} \) as \( G_{a1} \). This game represents the game that both stations share the node \( a \). The following two nonlinear programs to find the best response for two stations in this game.

\[
\max_{p_1, \lambda_{a1}} \pi_1 = p_1 \lambda_{a1} \tag{4.31}
\]

s.t. \( p_1 > 0, 0 \leq \lambda_{a1} \leq \Lambda_a, \lambda_{a1} < \mu \).
\[
\max_{p_2, \lambda_{a1}, \lambda_{b2}, y} \pi_2 = p_2(\lambda_{b2} + \lambda_{a2}) \tag{4.32}
\]

s.t. \( p_2 > 0, \lambda_{a2} + \lambda_{b2} < \mu. \)

Both programs satisfy the following constraints.

\[
\lambda_{a1} + \lambda_{a2} = \Lambda_a, \tag{4.33}
\]
\[
R_{a2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})} = R_{a1} - p_1 - \frac{h}{\mu - \lambda_{a1}} \geq v, \tag{4.34}
\]
\[
R_{b2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})} - v \geq -M(1 - y), \tag{4.35}
\]
\[
R_{b2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})} - v \leq My, \tag{4.36}
\]
\[
0 \leq \lambda_{b2} \leq y \Lambda_b, \quad y = \{0, 1\},
\]

where the binary variable \( y = 1 \{\lambda_{b2} > 0\} \), which represents the station 2’s decision on whether to attract customers at node \( b \). The number \( M > 0 \) is chosen with large enough value. The constraint (4.33) is present because of Assumption 9.

The problem (4.31) and (4.32) cannot be converted to the NCP as we did in solving the game \( G_a \) because of the binary decision variable \( y \). Fortunately, there is only one binary variable \( y \), the problem can be solved separately for each scenario in which the variable \( y = 0 \) or \( y = 1 \). Denote the solution to each scenario as \( p^*_1|y=j, p^*_2|y=j \) for \( j = 0, 1 \).

The solution in each scenario can be viewed as a candidate of Nash equilibrium of the game \( G_a \). We need to verify that if each candidate is the Nash equilibrium of the game \( G_{a1} \) or not. As the variable \( y \) stands for the decision of the station 2, we can verify that whether \( p^*_2|y=0 \) or \( p^*_2|y=1 \) is the best response to corresponding strategy of station 1. For instance, we can check \( p^*_2|y=0 \) to see if it is the best response to \( p^*_1|y=0 \) by changing \( p^*_2|y=0 \) to a value such that the station 2 starts to attract customers from node \( b \), i.e. \( \lambda_{2b} > 0 \) and \( y = 1 \) while the constraint (4.34) is still kept. Then,
we compare the best utility obtained in that case. If higher utility is obtained by attracting customers from node $b$, then it can be verified that $p_{2|y=0}^*$ is not a best response to $p_{1|y=0}^*$. Then we know that $(p_{1|y=0}^*, p_{2|y=0}^*)$ is not a Nash equilibrium of the game $G_{a1}$. The results of this verification is provided in the Appendix.

Note that the Nash equilibrium of the game $G_{a1}$ in one of the two cases,

1. The Nash equilibrium with $y = 0$, which is associated with state $x \in X_1 = \{[0, \Lambda_a - \lambda_{a2}, \lambda_{a2}, 0, 0, 0]^T, 0 \leq \lambda_{a2} \leq \Lambda_a\}$. Both stations only attract customers at node $a$.

2. The Nash equilibrium with $y = 1$, which is associated with state $x \in X_2 = \{[0, \Lambda_a - \lambda_{a2}, \lambda_{a2}, 0, 0, \lambda_{b2}]^T, 0 \leq \lambda_{a2} \leq \Lambda_a, 0 < \lambda_{b2} \leq \Lambda_b\}$.

We illustrate the state set of case 1 in Figure 4.6 and case 2 in Figure 4.7. We use pie charts to represent the node $a$ and $b$. The dark blue area represents the market share of the station 1 and red area represents the market share of the station 2. The yellow area means that the customers choose the alternative and join neither of the
Given that the Nash equilibrium in the game $G_{a1}$ is found, we study the relationship between the Nash equilibrium of $G_{a1}$ and $G_{ab}$. Given that the Nash equilibrium of $G_{a1}$ exists in the case 1 or 2, we focus on the condition for this equilibrium is Nash equilibrium of $G_{ab}$.

**Theorem 4.3.5.** *(Sufficient conditions for case 1)* Under the Assumption 8 and with inequality (4.27), the following results hold.

1) Given that the following inequality holds,

$$R_{b2} - p^*_{2|y=0} - \frac{h}{\mu} < v,$$

the Nash equilibrium $(p^*_{1|y=0}, p^*_{2|y=0})$ of the game $G_{a1}$ in case I is Nash equilibrium of
the game $G_{ab}$ if the following conditions hold,

$$\pi_1(p^*_1|y=0, p^*_2|y=0) \geq \pi_1(p^*_1, p^*_2|y=0), \quad (4.38)$$

$$\pi_2(p^*_1|y=0, p^*_2|y=0) \geq \pi_2(p^*_1|y=0, p^*_2), \quad (4.39)$$

where $p^*_1$ is the optimal solution to the following nonlinear program,

$$\max_{p_1, \lambda_{b1}} \pi_1 = p_1(\lambda_{b1} + \Lambda_a), \quad (4.40)$$

subject to

$$R_{b1} - p_1 - \frac{h}{\mu - (\Lambda_a + \lambda_{b1})} \geq v,$$

$$0 \leq \lambda_{b1} \leq \Lambda_b,$$

$$p_1 > 0.$$

The price $p^*_2$ is the optimal solution to the following nonlinear program.

$$\max_{p_2, \lambda_{b2}} \pi_2 = p_2(\lambda_{b2} + \Lambda_a), \quad (4.41)$$

subject to

$$R_{a2} - p_2 - \frac{h}{\mu - (\Lambda_a + \lambda_{b2})} \geq R_{a1} - p^*_1|y=0 - \frac{h}{\mu},$$

$$R_{b2} - p_2 - \frac{h}{\mu - (\Lambda_a + \lambda_{b2})} - v \geq 0,$$

$$0 \leq \lambda_{b2} \leq \Lambda_b,$$

$$p_2 > 0, \quad \Lambda_a + \lambda_{b2} < \mu.$$ 

Note that we let the revenue $\pi_1(p^*_1, p^*_2|y=0) = 0$ if the problem (4.40) does not have a feasible solution. Also, $\pi_2(p^*_1|y=0, p^*_2) = 0$ if the problem (4.41) does not have a feasible solution.

2) If (4.37) does not hold, then the Nash equilibrium $(p^*_1|y=0, p^*_2|y=0)$ of the game $G_{a1}$ is Nash equilibrium of the game $G_{ab}$ if the condition (4.39) holds.

Remark 4.3.6. The nonlinear program (4.40) calculates the optimal price for station
1 when the station lowers the price to attract customers from node \( b \). Since the inequality (4.37) holds, station 2 cannot attract any customer from the node \( b \).

**Remark 4.3.7.** The nonlinear program (4.41) calculates the optimal price for station 2 when the station 2 lowers the price to attract customers from node \( b \). Station 1 cannot attract any customer from the node \( b \) in this case.

Both nonlinear programs will be revisited in the proofs in Section 4.4.2.

The following theorem presents the sufficient condition for the case 2.

**Theorem 4.3.8.** (Sufficient conditions for case 2) Under the Assumption 8 and the condition (4.27), the Nash equilibrium of the game \( G_{a1} (p^*_1|y=1, p^*_2|y=1) \) in case I is the equilibrium of the game \( G_{ab} \) if the following conditions hold,

\[
\pi_2(p^*_1|y=1, p^*_2|y=1) \geq \pi_2(p^*_1|y=1, p^*_2), \quad (4.42)
\]

The price \( p^*_2 \) in this case is the solution to the following nonlinear program,

\[
\begin{align*}
\max_{p_2, \lambda_{b2}} & \quad \pi_2 = p_2(\lambda_{b2} + \Lambda_a), \\
\text{s.t.} & \quad R_{b2} - p_2 - \frac{h}{\mu - (\Lambda_a + \lambda_{b2})} \geq \max\{R_{b1} - p^*_1|y=1 - \frac{h}{\mu}, v\}, \\
& \quad 0 \leq \lambda_{b2} \leq \Lambda_b, \\
& \quad p_2 > 0, \quad \Lambda_a + \lambda_{b2} < \mu.
\end{align*}
\]

4.4 Proofs of the Main Result

4.4.1 The Equilibrium at Node \( a \) with \( R_{a2} < R_{b2} \)

In this part, we prove the Theorem 4.3.2. The solution \( (p^c_{1,a}, p^c_{2,a}) \) is obtained in the feasible set \( F_a \subset F \). To verify the price profile \( (p^c_{1,a}, p^c_{2,a}) \) is a Nash equilibrium for the game \( G_{ab} \), we have to make sure that there is no other alternative price for each station outside the feasible set \( F_a \) to achieve a higher revenue while keeping the
other station’s price fixed. The alternative price is explored by unilaterally varying
one station’s price in order to change the state $x$ so that the triple $(p_1, p_2, x)$ may
be outside the feasible set $F_a$. The revenue associated with the triple in that case is
compared to the the revenue achieved in the Nash equilibrium of the game $G_a$.

The following supporting results are properties of the states when the price of one
of the stations is changed unilaterally.

**Proposition 4.4.1.** If the following inequality holds,

$$R_{b1} - \frac{h}{\mu - \Lambda_a} > R_{b2} - \frac{h}{\mu - \Lambda_b},$$  \hspace{1cm} (4.44)

then the station 1 is able to set a price $p_1 > 0$ that is low enough to attract customers
at the node $b$, with $p_2$ fixed at $p^c_{2,b}$.

Given the inequality (4.44) holds, then station 1 may be able to attract customers
at node $b$.

The best response for station 1 in $X_b$ is obtained by the following nonlinear pro-
gram.

$$\max_{p_{1,b}} \pi_1(p_{1,b}, p^c_{2,a}) = (\Lambda_a + \lambda)p_{1,b}$$  \hspace{1cm} (4.45)

\text{s.t. } R_{b1} - p_{1,b} - \frac{h}{\mu - (\Lambda_a + \lambda)} \geq v, \\
R_{b1} - p_{1,b} - \frac{h}{\mu - (\Lambda_a + \lambda)} = R_{b2} - p^c_{2,a} - \frac{h}{\mu - (\Lambda_b - \lambda)}, \\
p_{1,b} > 0, 0 \leq \lambda \leq \Lambda_b, \hspace{0.2cm} \Lambda_a + \lambda < \mu.

where the price $p_{1,b}$ is decision variable for station 1.

**Lemma 4.4.2.** Given the station 2’s price $p^c_{2,a}$, the maximum revenue $\pi^*_1(p_{1,b}, p^c_{2,a})$
in the nonlinear program (4.45) is strictly less than the revenue achieved at the Nash
equilibrium $\pi_1(p^*, p^*_2)$.
Proof. The revenue in (4.45) can be written as

$$\pi_1(p_{b1}, p_{b2}) = (\Lambda_a + \lambda) \left( R_{b1} - \frac{h}{\mu - (\Lambda_a + \lambda)} - (R_{b2} - p_{2,a}^* - \frac{h}{\mu - (\Lambda_a + \lambda)}) \right).$$

(4.46)

The derivative of the revenue (4.45) to $\lambda$ is

$$\frac{d\pi_1(p_{b1}, p_{b2}^c)}{d\lambda} = R_{b1} - R_{b2} + p_{2,a}^* - \frac{h\mu}{(\mu - (\Lambda_a + \lambda))^2} + \frac{h(\mu - (\Lambda_a + \Lambda_b))}{(\mu - (\Lambda_a + \lambda))^2}. \quad (4.47)$$

We will show this derivative is negative by considering the revenue of the station 1 as a function of $\Delta \lambda_1$, where $\Delta \lambda_1$ is defined as the change of the joining rate $\lambda_1$. This function can be written as

$$\pi_1(\lambda_1, \lambda_2, \Delta \lambda_1) = (\lambda_1 + \Delta \lambda_1) \left( R_{1a} - \frac{h}{\mu - \lambda_1} - \left( R_{2a} - p_{2,a}^* - \frac{h}{\mu - (\lambda_2 - \Delta \lambda_1)} \right) \right).$$

(4.48)

Consider the revenue at the Nash equilibrium of the game $G_a$, $\pi_1(\lambda_1^*, \lambda_2^*, \Delta \lambda_1)$. Taking partial derivative of (4.48) with respect to the variable $\Delta \lambda$, we obtain

$$\left. \frac{\partial \pi_1(\lambda_1^*, \lambda_2^*, \Delta \lambda)}{\partial \Delta \lambda} \right|_{\Delta \lambda=0} = R_{a1} - R_{a2} + p_{2,a}^c - \frac{h\mu}{(\mu - \lambda_1^*)^2} + \frac{h(\mu - (\Lambda_a + \Lambda_b))}{(\mu - \lambda_2^* - \Delta \lambda_1)^2} = 0. \quad (4.49)$$

This indicates that at the Nash equilibrium of the game $G_a$, the derivative is equal to zero. By Assumption 8 and the fact that $\lambda_1^* < \Lambda_a + \lambda$ and $\lambda_2^* > \Lambda_b - \lambda$, the following inequality is obtained

$$\frac{d\pi_1(p_{b1}, p_{b2}^c)}{d\lambda} < \left. \frac{\partial \pi_1(\lambda_1^*, \lambda_2^*, \Delta \lambda)}{\partial \Delta \lambda} \right|_{\Delta \lambda=0} = 0,$$

(4.50)

for $\lambda > 0$. This indicates that the maximum of the revenue in the state set $x \in X_b$
is taken at the state \([\Lambda_a, 0, 0, \Lambda_b]\) at the price \(\theta_1\). Note that \(\theta_1 < p_1^f\), which implies 
\[
\pi(\theta_1, p_{2,a}^c) < \pi(p_1^f, p_{2,a}^c).
\]
By the fact \(\pi(p_1^f, p_{2,a}^c) \geq \pi(p_1^f, p_{2,a}^c)\), we have \(\pi(\theta_1, p_{2,a}^c) < \pi(p_1^f, p_{2,a}^c)\).

\[\square\]

**Remark 4.4.3.** This implies that the revenue of station 1 satisfies \(\pi_1(p_1, p_{2,a}^c) \leq \pi_1(p_1^f, p_{2,a}^c)\) for \(0 \leq p_1 \leq p_1^f\). In other words, the revenue of station 1 achieved at the equilibrium cannot be increased by decreasing its price from \(p_1^f\) when the price of the station 2 is fixed at \(p_{2,a}^c\).

**Proposition 4.4.4.** If the following inequality holds,

\[
R_{b1} - p_{1,a}^c - \frac{h}{\mu - \Lambda_a} > v
\]

then there exists a threshold \(p_2^f = R_{b2} - \frac{h}{\mu - \Lambda_b} - (R_{b1} - p_{1,a}^c - \frac{h}{\mu - \Lambda_a})\) such that if the price of the station 2 lowers its price to \(p_2 \geq p_2^f\), the station 1 is able to attract customers at node \(b\).

The station 2 can select an optimal price \(p_{2,b}^r\) to maximize its revenue with states in the set \(x \in X_b\). This can be found by solving the following nonlinear program.

\[
\max_{p_{2,b}} \pi_2(p_{1,a}^c, p_{2,b}) = (\Lambda_b - \lambda) p_{2,b} \quad (4.52)
\]

s.t. \(R_{b2} - p_{2,b} - \frac{h}{\mu - (\Lambda_b - \lambda)} \geq v\),

\[
R_{b1} - p_{1,a}^c - \frac{h}{\mu - (\Lambda_a + \lambda)} = R_{b2} - p_{2,b} - \frac{h}{\mu - (\Lambda_b - \lambda)},
\]

\(p_{2,b} > 0, \quad 0 \leq \lambda \leq \Lambda_b, \quad \Lambda_a + \lambda < \mu\).

The solution to this optimization problem is denoted as \(\pi_2(p_{1,a}^c, p_{2,b}^r)\).

Now, we are ready to prove Theorem 4.3.2.
Proof. To show the pair \((p_{1,a}^c, p_{2,a}^c)\) is a Nash equilibrium, we need to prove that under the sufficient condition, the price \(p_{1,a}^c\) is the best response to \(p_{2,a}^c\), no matter how station 1 adjusts its price. The same is for station 2. We first argue that given the station 2’s price \(p_{2,a}^c\), the station 1’s best response is \(p_{1,a}^c\) in all available strategy set. The station 1 cannot be better off by increasing the price because if station 1 continue to increase the price, its customers will switch to the station 2. Since \((p_1, p_2, x)\) is still in the set \(\mathcal{F}_a\), any price higher than \(p_{1,a}^c\) cannot achieve a higher revenue. By Lemma 4.4.2, the station 1 cannot achieve a better price by decreasing its price to attract customers from node b. Therefore, the best response of the station 1 is \(p_{1,a}^c\).

Next, we argue that the station 2’s best response to the price of station 1 \(p_{1,a}^c\) is \(p_{2,a}^c\) given the condition (4.24) holds. The station 2 can either lower or raise prices. If station 2 lowers the price, then more customer at node a will join station 2 and \((p_1, p_2, x)\) is still in the set \(\mathcal{F}_a\), which cannot achieve a higher revenue. Then, consider that station 2 raises price above \(p_{2,a}^c\). Given that (4.37) holds, station 1 cannot attract customers from node b no matter how station 2 adjusts his price. This means that station 2 cannot change the state set from \(\mathcal{X}_a\) by raising price either. Thus, the best response for station 2 is still \(p_{2,a}^c\). Thus, the first part of Theorem 4.3.2 is proved.

If (4.37) does not hold, station 2 is able to raise the price \(p_2 \geq p_2^f\) to let station 1 attract customers from node b by Proposition 4.4.4. It is also easy to check that it is the only way to change the state set from being \(\mathcal{X}_a\). By doing this, the state is changed to \(x \in \mathcal{X}_b\). The station 2 achieves the highest revenue for \(x \in \mathcal{X}_b\) by solving the nonlinear program (4.24).

However, by condition (4.24), the station 2 cannot earn more by this strategy, so the best response is still \(p_{2,a}^c\).□

By symmetry, Theorem 4.3.2 can be extended to the cases with equilibrium formed at node b, such as the case \(\case{3}\) with an equilibrium formed at the node b and the case \(\case{1}\) with an equilibrium formed at the node b. The argument remains the same by
exchanging the role of the node $a$ and $b$.

4.4.2 The Equilibrium at Node $a$ in the Case (1)

In this section, we prove the Theorem 4.3.5. The following results are needed for the proofs.

The following proposition characterizes the state set in reaction to the price change of station 2.

**Proposition 4.4.5.** In both case 1 and 2, suppose the station 2 sets the price lower than the threshold $p_2 < R_{a1} - p_{1y=i}^* - \frac{h}{\mu} - (R_{a2} - \frac{h}{\mu - \lambda_a})$. With the price of station 1 fixed at $p_{1y=i}^*$, $i = 0, 1$, station 2 captures all customers at node $a$. In this situation, station 1 cannot attract customers from node $b$. i.e. $\lambda_{b1} = 0$. The state $x \in [0, 0, \Lambda_a, 0, 0, \lambda], 0 \leq \lambda \leq \Lambda_b$.

**Proof.** We prove by contradiction. Suppose station 1 can attract customers from node $b$ and so does station 2. This means that both stations form an equilibrium at node $b$, then the following constraint is satisfied,

$$R_{b1} - p_1 - \frac{h}{\mu - \lambda_{b1}} = R_{b2} - p_2 - \frac{h}{\mu - (\Lambda_a + \lambda_{b2})} \geq v, \quad (4.53)$$

$$\lambda_{b1} > 0, \lambda_{b2} > 0, \lambda_{b1} + \lambda_{b2} \leq \Lambda_b.$$ 

However, by Assumption, the following inequality holds

$$R_{a1} - p_1 - \frac{h}{\mu - \lambda_{b1}} - v > R_{a2} - p_2 - \frac{h}{\mu - (\Lambda_a + \lambda_{b2})} - v. \quad (4.54)$$

This means that station 1 still has higher valuation than station 2 at node $a$, which contradicts the fact that station 2 attracts all customers at node $a$. Therefore, once station 1 has customers joining rate $\lambda_{b1} > 0$ from node $b$, the station 2 will not attract any customers from this node. Thus, it is possible that the state related to
this scenario is \([0, 0, \Lambda_a, 0, \lambda_{b1}, 0]^T\). For station 2, the best response, denoted as \(p_2^*\), satisfies

\[
v < R_{a1} - p_1 - \frac{h}{\mu - \lambda_{b1}} = R_{a2} - p_2 - \frac{h}{\mu - \Lambda_a}.
\]  

(4.55)

However, by Assumption \(8\) this implies that

\[
v = R_{b1} - p_1 - \frac{h}{\mu - \lambda_{b1}} < R_{b2} - p_2 - \frac{h}{\mu - \Lambda_a},
\]  

(4.56)

which means that station 2 has even higher valuation for customers at node \(b\), this leads to a contradiction that \(\lambda_{b2} = 0\). Therefore, station 1 is impossible to attract any customer from node \(b\), given \(\lambda_{a2} = \Lambda_a\). \(\square\)

Remark 4.4.6. Proposition 4.4.5 considers the case that when the station 2 sets a low enough price to take all customers at node \(a\). The station 1, with zero congestion cost, may be possible to attract customers at node \(b\). However, according to Proposition 4.4.5 this case never happens.

Lemma 4.4.7. Given that the condition (4.37) does not hold, the station 1 cannot achieve a higher revenue by attracting customers from the node \(b\) than the revenue achieved at the price \(p_{1|y=0}^*\) in case 1 and \(p_{1|y=1}^*\) in case 2. In other words, the following inequality holds.

\[
\pi_1(p_1^*, p_2^*) \geq \pi_1(p_1, p_2^*),
\]  

(4.57)

where \(p_1^*, p_2^*\) denote \(p_{1|y=0}^*, p_{2|y=0}^*\) in case 1 and \(p_{1|y=1}^*, p_{2|y=1}^*\) in case 2. The price \(p_1 \neq p_1^*\) is the price of station 1 that attracts customers at node \(b\).

Proof. We first consider the case 1, where the station 2 does not attract customers from the node \(b\). It is obvious that when the station 1 lowers its price, the customers
at node \(a\) will leave station 2. If the condition \([4.37]\) does not hold, then station 2 is able to attract customers at node \(b\) when all customers at the node \(a\) leave, \(i.e. \lambda_2 = 0\). Suppose in this case, only part of the customers at the node \(b\) are attracted to station 1 or 2, \(i.e. \lambda_{b1} > 0, \lambda_{b2} > 0\). For station 2, the following equality holds,

\[
R_{b2} - p_{2|y=0}^* - \frac{h}{\mu - \lambda_2} = v, \tag{4.58}
\]

By the fact that \(R_{a2} > R_{b2}\), it is easy to see that \(\lambda_2 < \lambda_2^*\), where \(\lambda_2^*\) denotes the joining rate at the Nash equilibrium of the game \(G_a\).

Denote the joining rate of station 1 at the node \(b\) as \(\lambda_{b1}\). The revenue of station 1 when station 1 attracts customers at node \(b\) given that \((4.58)\) holds can be written as follows,

\[
\pi_1(p_{b1}, p_{2|y=0}^*) = p_1(\Lambda_a + \lambda_{b1})
\]

\[
= (R_{b1} - \frac{h}{(\mu - (\Lambda_a + \lambda_{b1}))} - R_{b2} + p_{2|y=0}^* + \frac{h}{\mu - \lambda_2})(\Lambda_a + \lambda_{b1}).
\]

The derivative of the revenue is

\[
\frac{d\pi_1(p_{b1}, p_{2|y=0}^*)}{d\lambda_{b1}} = R_{b1} - R_{b2} + p_{2|y=0}^* - \frac{h\mu}{(\mu - (\Lambda_a + \lambda_{b1}))^2} + \frac{h}{\mu - \lambda_2}. \tag{4.59}
\]

By definition of the Nash equilibrium of the game \(G_{a1}\), the derivative of the joining rate \(\lambda_{a1}\) at the \(\lambda_{a1} = \lambda_1^*\) satisfies

\[
\frac{d\pi_1}{d\lambda_1}\bigg|_{\lambda_1 = \lambda_1^*} = R_{a1} - R_{a2} - \frac{h\mu}{(\mu - \lambda_1)^2} + p_{2|y=0}^* + \frac{h}{\mu - \lambda_2} = 0. \tag{4.60}
\]

Compare the value of two derivatives and notice that \(R_{a1} - R_{a2} > R_{b1} - R_{b2}\) and
\( \lambda^*_1 < \Lambda_a + \lambda_{b1} \), the following inequality holds,

\[
\frac{d\pi_1(p_{b1}, p_{2|y=0}^*)}{d\lambda_{b1}} < \frac{d\pi_1}{d\lambda_1} \bigg|_{\lambda_1=\lambda^*_1} = 0.
\] (4.61)

This means that the derivative of the revenue with respect to \( \lambda_{b1} \) is negative. Moreover, if the price of station 1 is even lower and share with station 2 at node \( b \) with the utility greater than zero, which implies that the joining rate of station 2 \( \lambda_2 = \Lambda_b - \lambda_a < \lambda^*_2 \). In this case, the revenue of station 1 can be obtained as

\[
\pi_1(p_{b1}, p_{2|y=0}^*) = p_1(\Lambda_a + \lambda_{b1}) \\
= R_{b1} - \frac{h}{(\mu - (\Lambda_a + \lambda_{b1}))} - R_{b2} + p_{2|y=0}^* + \frac{h}{\mu - (\Lambda_b - \lambda_{b1})}.
\]

The derivative of the revenue is

\[
\frac{d\pi_1(p_{b1}, p_{2|y=0}^*)}{d\lambda_{b1}} = R_{b1} - R_{b2} + p_{2}^* - \frac{h\mu}{(\mu - (\Lambda_a + \lambda_{b1}))^2} + \frac{h}{\mu - (\Lambda_b - \lambda_{b1})} \\
- \frac{h(\Lambda_a + \lambda_{b1})}{(\mu - (\Lambda_b - \lambda_{b1}))^2}.
\] (4.62)

Notice that \( R_{a1} - R_{a2} > R_{b1} - R_{b2} \), \( \Lambda_a + \lambda_{b1} > \lambda^*_1 \) and \( \Lambda_b - \lambda_{b1} < \lambda^*_2 \), the condition (4.61) still holds.

Thus, the highest revenue that can be achieved in the case where \( \lambda_{b1} \geq 0 \) is actually \( \lambda_{b1} = 0 \). The revenue with \( \lambda_{b1} = 0 \) reaches the highest value when the price of station 1 satisfies

\[
R_{a1} - p_1 - \frac{h}{\mu - \Lambda_a} = v.
\] (4.63)

However, the revenue is no higher than the revenue \( \pi_1(p_{1|y=0}^*, p_{2|y=0}^*) \) at the Nash equilibrium of \( G_{a1} \).

It is easy to check that this result also holds for the case 2. The same argument
holds when $p_{2|y=0}^*$ is replaced by $p_{2|y=1}^*$ and $\lambda_2 = \lambda_2^*$ in the (4.58).

Now, we are ready to prove Theorem 4.3.5.

Proof of Theorem 4.3.5

Let us first consider the alternative for station 2 while keeping the station 1’s price at $p_{1|y=0}^*$. If the price $p_2$ is increased, the state is still in the state set $\mathcal{X}_1$, the best response is $p_{2|y=0}^*$. If station 2 lower its price, the rate $\lambda_{a1}$ will decrease and the rate $\lambda_{a2}$ will increase. When the station 2 set a price low enough such that $\lambda_{a2} = \Lambda_{a}$ and $\lambda_{b2} > 0$. By Proposition 4.4.5 we know that $\lambda_{b1} = 0$, i.e. station 1 cannot attract customers at node b. The state set in this case can be determined as $\{[0, \Lambda_{a}, 0, \lambda]^T, 0 \leq \lambda \leq \Lambda_{b}\}$. The optimal price for station 2 with this state set can be obtained by the nonlinear program (4.41), which is $p_r^e$. By the condition (4.39), we know that the revenue achieved by the price $p_{2|y=0}^*$ is greater than $p_r^e$. Thus, the best response of station 2 is $p_{2|y=0}^*$.

The station 1’s response to $p_2^*$ can be considered as follows. The station 1 cannot increase the price because the state set is still $\mathcal{X}_1$ and this cannot lead to a better revenue.

If the station 1 lowers the price, then station 2 will start to lose customers from node a. By lowering the price below a threshold, station 1 takes all customers at node a. At this point, two scenarios may happen. The first scenario is that the following inequality holds,

$$R_{b2} - p_{2}^* - \frac{h}{\mu} > v.$$  (4.64)

The inequality (4.64) implies that station 2 can still attract customers from node $b$, i.e. $\lambda_{b2} > 0$ because station 2 loses customers at node $a$ and the waiting cost is reduced. In this scenario, station 1 has two choices. The first is that the station 1 does not lower the price to attract customers at node $b$, the corresponding state will
be in the state set $\mathcal{X}_2$. Then the best response for station 1 $p'_1$ has to satisfy

$$R_{a1} - p'_1 - \frac{h}{\mu - \Lambda_a} = R_{a2} - p^*_2|_{y=0} - \frac{h}{\mu - \lambda_{b2}}. \quad (4.65)$$

This is because the joining rate for station 1 is fixed at $\Lambda_a$ and the station 1 cannot lower the price than $p'_1$ to obtain a better revenue. Still, the triple $(p_1, p_2, x) \in \mathcal{F}_2$. Therefore, any response for station 1 is still in the sense of game $\mathcal{G}_{a1}$. By the definition of Nash equilibrium in case 1, the best response to $p^*_2|_{y=0}$ is $p^*_1|_{y=0}$.

The second choice for station 1 is to form a market equilibrium with station 2 at node $b$. However, Lemma 4.4.7 implies that station 1 cannot achieve a better revenue than $p^*_1|_{y=0}$. The other scenario is that the condition (4.64) does not hold, which means station 2 cannot attract customers from node $b$. The station 1 also has to decide whether or not attract customers at node $b$. If station 1 does not attract customers from node $b$, then the scenario falls into the setting of the game $\mathcal{G}_{a1}$ and the best response is $p^*_1|_{y=0}$. If the station 1 attracts customers from the node $b$, then best response $p'_1$ in this case can be obtained by the nonlinear program (4.40).

All the possible scenarios in the calculation of the best response of station 1 is summarized in Table 4.3. By Table 4.3, we can see that if the condition (4.37) holds, i.e. $\lambda_{b2} = 0$, then the best response to station 1 is

$$\arg\max_{p_1} \{ \pi_1(p_1 = p^*_1|_{y=0}, p^*_2|_{y=0}), \pi_1(p_1 = p'_1, p^*_2|_{y=0}) \}.$$ 

By the condition (4.38), the station 1’s best response is $p^*_1|_{y=0}$. Since the best response for station 2 is $p^*_2|_{y=0}$, we prove the first part of Theorem 4.3.5.

If the condition (4.37) does not hold, then the station 1’s best response is $p^*_1|_{y=0}$. Since the best response for station 2 is $p^*_2|_{y=0}$, the second part of Theorem 4.3.5 holds.

Proof of Theorem 4.3.8
The best response for station 1 in the case 1

<table>
<thead>
<tr>
<th>Condition</th>
<th>( \lambda_{b2} &gt; 0 )</th>
<th>( \lambda_{b2} = 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_{a1} &gt; 0 )</td>
<td>( p_1^{*}</td>
<td>_{y=0} )</td>
</tr>
<tr>
<td>( \lambda_{a1} = 0 )</td>
<td>( p_1^{*}</td>
<td>_{y=0} )</td>
</tr>
</tbody>
</table>

We consider all possible deviations from \( p_2^{*}|_{y=1} \) that the station 2 can make to achieve a higher utility than \( p_2^{*}|_{y=1} \) in the case (II). The station 2 can either lower first, station 2 cannot increase the price to obtain a higher revenue because the feasible set is still \( F_2 \) of the game \( G_{a1} \).

If station 2 lowers the price to capture the entire node \( a \) and check if this leads to a higher revenue. When station 2 adjusts the price to be low enough to capture all customers at node \( a \), the station 1 cannot attract customers from node \( b \) by proposition 4.4.5. Thus, we have \( \lambda_{a1} = \lambda_{b1} = 0 \). Thus, the station 2 will take all customers at \( a \) and also be able to attract customers from node \( b \). In this case, the best response \( p_2^{*} \) for station 2 can be obtained by the following nonlinear program (4.43). If the condition (4.42) holds, then we know that \( p_2^{*} \) cannot achieve a higher utility than the utility achieved by \( p_2^{*}|_{y=1} \). Summarizing all alternatives of the station 2, we can see that if condition (4.42) holds, the best response for station 2 is \( p_2^{*}|_{y=1} \).

Next, we consider the alternatives for station 1 in the case 2. Station 1 cannot increase the price to obtain a higher revenue because the feasible set is still \( F_2 \) of the game \( G_a \).

If station 1 lowers the price such that only attracts all customers from the node \( a \), but does not attract any customers from node \( b \), then the optimal price for station...
1 is $p_1^*$ that satisfies,

$$R_{a1} - p_1^* - \frac{h}{\mu - \Lambda_a} = R_{a2} - p_{2|y=1}^* - \frac{h}{\mu - \lambda_{b2}}. \tag{4.66}$$

However, the feasible set is still $\mathcal{F}_2$ of the game $G_{a1}$. And we already know that the best response for station 1 is $p_{1|y=1}^*$.

The other alternative for station 1 is lower the price to attract customers at node $b$ and form an equilibrium with station 2 at that node. The best response in this case is $p_{1|y=1}^*$ according to Lemma 4.4.7.

To summarize, the best response for station 1 is $p_{1|y=1}^*$. Since the best response for station 2 is $p_{2|y=1}^*$.

Therefore, if the condition (4.42) holds, we know that both prices in $(p_{1|y=1}^*, p_{2|y=1}^*)$ are best responses, then the price profile $(p_{1|y=1}^*, p_{2|y=1}^*)$ is a Nash equilibrium of game $G_{ab}$.

4.5 Numerical Examples

Example 4 (Two stations share node $a$ in scenario 3). Consider two stations in a layout of the scenario 3. The constant for waiting is $h = 1$, the alternative value $v = 1$. The service rate $\mu = 6$, the valuations of customer at node $i$ selecting station $j$ are given as $R_{a1} = 5.4$, $R_{b1} = 3.4$, $R_{a2} = 5.4$ and $R_{b2} = 8$. The demand generating rates $\Lambda_a = 3.9$ and $\Lambda_b = 2$.

The market equilibrium at the node a can be computed according to nonlinear programs (4.20) and (4.21), and the total customer joining rate of the station 1 is $\lambda_1 = 1.095$. For station 2, the joining rate is $\lambda_2 = 4.805$. The service price for station 1 is $p_1 = 1.095$. The service price for station 1 is $p_2 = 3.563$. To verify that the profile $(p_1 = 1.095, p_2 = 3.563)$ is the Nash equilibrium of the game $G_{ab}$, the nonlinear program (4.24) is considered. It turns out that there is no feasible
solution to this nonlinear program. Thus, according to Theorem 4.3.2, the price profile \((p_1 = 1.095, p_2 = 3.563)\) is the Nash equilibrium of the game \(G_{ab}\).

**Example 5** (Two stations share node \(a\) in scenario \(\mathcal{1}: 1\)). Consider two stations in a layout of the scenario \(\mathcal{1}\). The constant for waiting is \(h = 1\), the alternative value \(v = 1\). The service rate \(\mu = 6\), the valuations of customer at node \(i\) selecting station \(j\) are given as \(R_{a1} = 3\), \(R_{b1} = 1\), \(R_{a2} = 3.2\) and \(R_{b2} = 1.3\). The demand generating rates \(\Lambda_a = 3.9\) and \(\Lambda_b = 1.5\).

The Nash equilibrium of the game \(G_{a1}\) can be computed according to nonlinear programs (4.20) and (4.21) within the feasible set \(F_1\) \((y = 0, \text{Case 1})\). At that equilibrium, the total customer joining rate of the station 1 is \(\lambda_1 = 1.424\). For station 2, the joining rate is \(\lambda_2 = 2.476\). The service price for station 1 is \(p_1 = 0.183\). The service price for station 1 is \(p_2 = 0.318\). Using the condition (4.67), we can verify that the profile \((p_1 = 0.183, p_2 = 0.318)\) is the Nash equilibrium of the game \(G_{a1}\). This is due to the fact that the difference of the distances between the station 2 to node \(a\) and node \(b\) is relatively large. To verify this is the Nash equilibrium of the game \(G_{ab}\), we compute the nonlinear program (4.40) and the nonlinear program (4.41). It turns out that there is no feasible solution to both nonlinear programs. Thus, we conclude the price profile \((p_1 = 0.183, p_2 = 0.318)\) is the Nash equilibrium of the game \(G_{ab}\).

**Example 6** (Two stations share node \(a\) in scenario \(\mathcal{1}: II\)). Consider two stations in a layout of the scenario \(\mathcal{1}\). The constant for waiting is \(h = 1\), the alternative value \(v = 1\). The service rate \(\mu = 6\), the valuations of customer at node \(i\) selecting station \(j\) are given as \(R_{a1} = 3\), \(R_{b1} = 2\), \(R_{a2} = 3.2\) and \(R_{b2} = 2.3\). The demand generating rates \(\Lambda_a = 3.9\) and \(\Lambda_b = 1.5\).

The Nash equilibrium of the game \(G_{a1}\) is computed according to nonlinear programs (4.20) and (4.21) within the feasible set \(F_2\) \((y = 1, \text{Case 2})\). At that equilibrium, the total customer joining rate of the station 1 is \(\lambda_{a1} = 2.633\). For station 2, the joining rate is \(\lambda_{a2} = 2.476, \lambda_{b2} = 2.585\). The service price for station 1 is \(p_{1|y=1} = 0.803\).
The service price for station 1 is $p_{2|y=1}^* = 0.835$. Using the condition (4.77), we can verify that the profile $(p_{1|y=1}^* = 0.803, p_{2|y=1}^* = 0.835)$ is the Nash equilibrium of the game $G_{a1}$.

To verify this is the Nash equilibrium of the game $G_{ab}$, we compute the nonlinear program (4.43). The result is $p_2^* = 0.793, \lambda_{y2} = 0, \lambda_{a2} = \Lambda_a$, the revenue achieved is 3.0927 which is less than 3.2164 the revenue achieved at $p_{2|y=1}^*$. Thus, the price profile $(p_{1|y=1}^* = 0.803, p_{2|y=1}^* = 0.835)$ is the Nash equilibrium of the game $G_{ab}$.

4.6 Conclusions

In this chapter, we consider the strategy for two competitive, selfish service stations that maximize the revenue over the price and customer’s joining rates. The two service stations are viewed as two players in a game with individual payoff defined as revenue and price as strategy. Both stations have to be aware of the other station’s strategy and also the response of the customers at different locations. Two nonlinear programs are formulated to compute the price profile for best response within a given set of customer assignments. The constraint of the nonlinear programs varies in different scenarios in which both station form a market equilibrium at different customer locations. Sufficient conditions for a given set of strategies being the Nash equilibrium are obtained.

4.7 Appendix

In this appendix, we present the result about verification of the Nash equilibrium of the game $G_{a1}$.

The following result is used to verify that $p_{2|y=0}^*$ is the best response to $p_{1|y=0}$, with $y = 0$ in the game $G_{a1}$.

**Proposition 4.7.1.** Consider the solution $(p_{1|y=0}, p_{2|y=0})$ to the problem (4.31) and
The price $p^*_{2|y=0}$ is the best response to $p^*_{1|y=0}$ if the following condition holds,

$$R_{a1} - p^*_{1|y=0} - \frac{h}{\mu - \lambda_{a1} - \lambda_{b2}} < R_{a2} - R_{b2}. \quad (4.67)$$

Proof. Suppose the station 2 adjust the price to $p_2$ such that the station 2 is able to attract customers from node $b$. It is easy to see that the utility of the customer at node $b$ joins station 2 will satisfy the following equation.

$$v = R_{b2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})}. \quad (4.68)$$

where $0 < \lambda_{b2} \leq \Lambda_b$. At the same time, the station 1 and 2 must share the node $a$, the following equation holds.

$$v_1 = R_{a1} - p^*_{1|y=0} - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} = R_{a2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})}. \quad (4.69)$$

Note that the value of $v_1$, which is the utility of the customers at node $a$. The difference between $v_1$ and $v$ can be obtained as

$$v_1 - v = R_{a1} - p^*_{1|y=0} - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} - v = R_{a2} - R_{b2} \quad (4.70)$$

Note that $v_1$ takes the maximum value when $\lambda_{a2} = \Lambda_a$ and takes the minimum value when $\lambda_{a2} = 0$. In that case, we have

$$R_{a1} - p^*_{1|y=0} - \frac{h}{\mu} - v \geq R_{a2} - R_{b2}. \quad (4.71)$$

This is a necessary condition for the station 2 to be able to attract customers at node $b$ while keeping the constraint (4.69) which ensures that both stations share the node $a$. Thus, if the condition (4.67) holds, there is no other alternatives for station 2 to deviate from the $y = 0$ to $y = 1$, which means that $p^*_{2|y=0}$ is the best response to
Remark 4.7.2. The condition (4.67) actually provides a threshold for the difference $R_{a2} - R_{b2}$ to determine whether the station 2 is able to attract the customer from node $b$. This is related to the location of the node $b$. Since in the case $\textcircled{1}$, the node $b$ is relatively far away from both stations than node $a$, then it is possible that the node $b$ is too far away for station 2 to attract customers from that node.

Further, we also consider case that (4.67) does not hold, in which case the station 2 can attract customers from node $b$ and see if he can get better utility. We also obtain a result in that situation. Before we state the result, the following nonlinear program is defined as follows.

$$\begin{align*}
\max_{p_2, \lambda_{a2}, \lambda_{b2}} \pi_2(p_{1|y=0}^*, p_2) &= p_2(\lambda_{a2} + \lambda_{b2}) \\
\text{s.t.} \quad R_{a1} - p_{1|y=0}^* - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} &= R_{a2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})} \geq v, \\
R_{b2} - p_2 - \frac{h}{\mu - (\lambda_{a2} + \lambda_{b2})} &= v. \\
0 < \lambda_{a2} < \Lambda_a, 0 < \lambda_{b2} < \Lambda_b.
\end{align*}$$

(4.72)

(4.73)

(4.74)

(4.75)

Proposition 4.7.3. Given that the condition (4.67) does not hold, the price $p_{2|y=0}^*$ is the best response to $p_{1|y=0}^*$ if the following condition holds,

$$\pi_2(p_{1|y=0}^*, p_{2,0}^*) \leq \pi_2(p_{1|y=0}^*, p_{2|y=0}^*).$$

(4.76)

where $p_{2,0}^*$ is the optimal solution to the problem (4.72).

Proof. If the condition (4.67) does not hold, then it means that the station 2 can attract the customers from node $b$. Thus, an alternative price the station 2 can choose is the optimal solution $p_{2,0}^*$. If (4.76) holds, meaning that the station 2 cannot get a higher utility by choosing $p_{2,0}^*$, then it implies that the best response is $p_{2|y=0}^*$.

\[\square\]
Similarly, we consider the case 2 to verify the $p^*_2|y=1$ is the best response to the price $p^*_1|y=1$.

**Proposition 4.7.4.** Consider the solution $(p^*_1|y=1, p^*_2|y=1)$ to the problem (4.31) and (4.32). The price $p^*_2|y=1$ is the best response to $p^*_1|y=1$ if the following condition holds,

$$R_{a1} - p^*_1|y=1 - \frac{h}{\mu - \Lambda_a} - v > R_{a2} - R_{b2}.$$  (4.77)

**Proof.** Suppose the station 2 adjusts the price to $p_2$ such that the station 2 is able to abandon customers from node $b$. It is easy to see that the utility of the customer at node $b$ leave station 2 will satisfy the following equation.

$$v > R_{b2} - p_2 - \frac{h}{\mu - \lambda_{a2}}.$$  (4.78)

where $0 < \lambda_{b2} \leq \Lambda_b$. At the same time, the station 1 and 2 share the node $a$, the following equation holds.

$$v_2 = R_{a1} - p^*_1|y=1 - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_{a2}}.$$  (4.79)

Note that the value of $v_2$, which is the utility of the customers at node $a$. The difference between $v_2$ and $v$ can be obtained as

$$v_2 - v = R_{a1} - p^*_1|y=1 - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} - v < R_{a2} - R_{b2}.$$  (4.80)

Note that in order to let the state vector $x$ remains in the feasible set $\mathcal{F}_1$, then there must exist a $0 < \lambda_{a2} < \Lambda_a$ such that (4.78) and (4.79) hold simultaneously. This becomes impossible if $v_2$ takes the minimum value when $\lambda_{a2} = 0$ and $R_{a2} - R_{b2} <$
\[
\min v_2. \text{ In that case, we have}
\]
\[
R_{a1} - p_{1|y=1}^* - \frac{h}{\mu - \Lambda_a} - v > R_{a2} - R_{b2}. \tag{4.81}
\]

Thus, if the condition (4.77) holds, there is no other alternatives for station 2 to deviate from the \( y = 1 \) to \( y = 0 \), which means that \( p_{2|y=1}^* \) is the best response to \( p_{1|y=1}^* \).

Similarly, if the condition (4.77) does not hold, then station 2 is possible to let the customers from node \( b \) leave and seek alternative price that may lead to a higher utility. The following nonlinear program can be used to check whether this alternative can let station 2 be better off.

\[
\max_{p_2, \lambda_{a2}} \pi_2(p_{1|y=1}^*, p_2) = p_2 \lambda_{a2} \tag{4.82}
\]
\[
\text{s.t. } R_{a1} - p_{1|y=1}^* - \frac{h}{\mu - (\Lambda_a - \lambda_{a2})} = R_{a2} - p_2 - \frac{h}{\mu - \lambda_{a2}} \geq v, \tag{4.83}
\]
\[
R_{b2} - p_2 - \frac{h}{\mu - \lambda_{a2}} < v. \tag{4.84}
\]
\[
0 < \lambda_{a2} < \Lambda_a. \tag{4.85}
\]

**Proposition 4.7.5.** Given that the condition (4.77) does not hold, the price \( p_{2|y=0}^* \) is the best response to \( p_{1|y=1}^* \) if the following condition holds,

\[
\pi_2(p_{1|y=1}^*, p_{2|y=1}^*) \leq \pi_2(p_{1|y=1}^*, p_{2|y=0}^*). \tag{4.86}
\]

where \( p_{2|y=1}^* \) is the optimal solution to the problem (4.82).

**Proof.** The proof is omitted since it is similar to the proof of the Proposition 4.7.3.
CHAPTER 5

OPTIMAL CHARGING PROFILES AND PRICING STRATEGIES FOR ELECTRIC VEHICLE CHARGING STATIONS

5.1 Introduction

Plug-in electric vehicles (PEVs) are expected to have a main role on the next-generation of vehicles. PEVs run only on battery power, which is free of fossil fuels and leads to lesser pollution. However, the adoption of PEVs relies on the infrastructure of battery charging. It is very possible that high penetration of PEVs will stimulate new charging station business [33]. In the commercial charging business, charging station owners will provide charging services to PEV drivers at a certain price and a particular charging rate profile. An optimal operation of charging station is desired to manage the charging profile and create appropriate economic incentives in order to maximize profits while keeps the customers satisfied.

In this chapter, we view the station and the customer as market players who will interact with the power price from the utilities and the price of charging. The desirable scenario is that both players reach an equilibrium of the market. At the equilibrium, both parties maximize their welfare and will not be better off by changing their strategy.

We start the analysis by assuming that the price of charging is formed by the electric vehicle charging market and therefore is exogenous. Thus, the price is independent of the players’ actions. Under this assumption, the market equilibrium is called a competitive equilibrium [4]. Thus, given the price of charging, the charging
profile at the competitive equilibrium is optimal for both station owner and the PEV customer.

What makes the problem more challenging is that the variables in traditional competitive equilibrium models are usually static [4]. In the charging station problem considered in this chapter, dynamic constraints of the battery—resulting from costs of cycling—are considered. The decision variables and the prices are trajectories defined over a finite time interval. Thus, the competitive equilibrium problem must be posed in the domain of functions. In [76], such equilibrium is modeled as a dynamic equilibrium, which can be tackled by using Lagrangian relaxation and variational calculus. This concept of dynamic equilibrium is utilized to analyze major wholesale power market players such as generators, utility companies as well as transmission owners.

Although this work is aligned with the framework of dynamic equilibrium [76], the main contribution is to consider the boundary value conditions and constraints in a finite horizon setting, which is close to the situation of electric vehicle charging. This is distinct from the model in [76], which studies the long term behavior of the wholesale power market in a infinite horizon. Specifically, we characterize the competitive equilibrium by applying optimal control model and Pontryagin’s minimum principle [29, 30, 78]. The trajectory of the price and charging rate are given at the equilibrium. Moreover, we show that the charging rate profile obtained at the competitive equilibrium is efficient, which is consistent with the result in [76].

The competitive equilibrium is first characterized in the scenario with a single customer and a station and then extended to multiple customers with predetermined arrival times. Each charging profile of the customer at the competitive equilibrium are obtained with consideration of the arrival times which maximize each customer’s utility. The aggregation of the charging profiles maximize the station’s utility.

Since the competitive equilibrium assumes that the price is exogenous, one inter-
esting point is to analyze the case in which the price can be influenced by the players. Sometimes, the charging station, as an enterprise, may have done extensive research on customer behavior and get access to the customer model. Moreover, the charging station can even affect the price. Suppose the charging station has the capability of setting the price and anticipating the customers reaction with respect to the price. Then the station can design a particular price profile and charging rate profile, such that the charging profile is optimal for customer under the price. This scenario fits the Stackelberg game model [18, 27], where the station is the leader and the customer is the follower. The leader sets up the strategies before the customer and then the customer can only accepts the decision of the leader since he cannot achieve higher utility by deviating from the decisions of the leader. In order to prevent the station from charging arbitrarily high price from the customer, a constraint on the minimum utility of the customer is imposed.

We investigate the Stackelberg equilibrium in the dynamic equilibrium framework, which is another contribution in this work. Various scenarios such as single customer and batch arrivals of the customers are also considered. The social welfare of the Stackelberg equilibrium is evaluated in this chapter. An upper bound of the gap of the social welfare between the Stackelberg and competitive equilibrium is obtained, which indicates that operating at Stackelberg equilibrium incurs a loss of the social welfare in most cases. This social welfare gap is also derived in the batch arrival case, with a certain assumption on the homogeneity of the customers.

The rest of the chapter is organized as follows. Section 5.2 presents the models of the charging stations and customers as market players. The individual optimization problem for each player is formulated. A social planner’s problem is provided in Section 5.3. Both single customer and multiple customers in a batch arrivals are addressed. Section 5.4 presents the main result on the competitive equilibrium. The result is first presented in a scenario with a single customer and then discussed in the
5.2 Charging Station and Customer Model

We consider the charging process within a time frame \([0, T]\), where \(T > 0\) is the length of the charging period. Denote the energy charged into the battery or the state of charge (SoC) as \(x(t)\). The profile of SoC is defined as \(x(t) \in C^2(0, T]\) and the charging rate is the time derivative \(\dot{x}(t)\). Denote the SoC level of a vehicle at the end of the charging period as \(X\) and clearly, \(x(T) = X\). In addition, we assume that the electricity price profile from the utility within this period is known a priori as \(r(t)\) for \(t \in [0, T]\).

5.2.1 The Charging Station Problem

The charging station wants to maximize his profit by choosing appropriate charging profile \(x_S(t)\) with respect to a given service price \(p(t)\) for \(t \in [0, T]\). The profit is written as

\[
\int_0^T (p(t) - r(t)) \dot{x}_S(t) dt. \tag{5.1}
\]

We assume that the charging profile satisfies the following constraints:

- The station is not allowed to sell power back to the grid at any time \(t\), which means that the charging rate must be non-negative at any time \(t\). For instance, if there is only a single customer, his vehicle cannot be discharged at any time \(t\), i.e. \(\dot{x}_S(t) \geq 0\).

- The initial SoC level and the final SoC level at the end of the charging is determined. In other words, the SoC satisfies the boundary conditions \(x(0) = x_0, x(T) = X\).

With those constraints, the charging station’s problem is given as
**Definition 5** (Charging Station’s problem). Given the initial SoC level $x_0 > 0$ and the final SoC level $X > x_0$, the charging station chooses charging rate profile under a certain constraint over the time horizon $[0, T]$ to maximize the utility function (5.1). The optimization problem is written as follows.

$$\max_{\dot{x}_S(t)} \int_0^T (p(t) - r(t)) \dot{x}_S(t) dt.$$ \hspace{1cm} (5.2)

Subject to $(x_S(t), \dot{x}_S(t)) \in X_S$.

where the feasible set $X_S = \{x_S(t), t \in [0, T] \mid \dot{x}_S(t) \geq 0, x_S(T) = X, x_S(0) = x_0\}$.

### 5.2.2 The PEV Problem

We assume that a customer receives utility $\alpha$ for a unit of power charged into the vehicle. The utility within an infinitesimal interval $dt$ is $\alpha \dot{x}(t) dt$. The customer pays the price $p(t)$ for the power of the amount $\dot{x}(t) dt$. Besides, we also consider the cost of battery health degradation, which is related to the charging rate. The cost is defined as $\beta \dot{x}^2(t)$ where $\beta > 0$ is a coefficient. With consideration of the above costs, the customer wishes to maximize his profit by choosing the charging profile $x_D(t)$ in terms of the price profile $p(t)$.

$$\int_0^T (\alpha - p(t)) \dot{x}_D(t) - \beta \dot{x}_D^2(t) dt.$$ \hspace{1cm} (5.3)

The charging rate that satisfies the PEV customer has the following constraints:

- For each vehicle, the initial SoC level $x(0) > 0$ is known. Each vehicle wants to charge the vehicle to the SoC level $x(T) = X$ in the end of the charging process.

- For the battery health, the battery cannot be discharged too deeply. The SoC at any time $t$ should be greater than a given level. For each vehicle $i$, it is required that $x_i(t) \geq L_i x_i(0)$ where $0 < L_i < 1$. Also, $X$ is chosen appropriately to avoid overcharging the battery.
The customer is interested in maximizing the utility under the given market price signal \( p(t) \). We define the customer’s problem as follows.

**Definition 6** (Customer’s problem). *Given the initial SoC level \( x_0 > 0 \) and the final SoC level \( X > x_0 \), the customer chooses charging rate profile under a certain constraint over the time horizon \([0, T]\) to maximize the utility function \((5.3)\). The optimization problem is written as follows.*

\[
\max_{\dot{x}_D(t)} \int_0^T (\alpha - p(t)) \dot{x}_D(t) - \beta \dot{x}_D^2(t) \, dt. \tag{5.4}
\]

\[\text{s.t. } (x_D(t), \dot{x}_D(t)) \in X_D\]

where \( X_D \) is the feasible set of \( x_D(t) \) and \( \dot{x}_D(t) \), which is given as \( X_D = \{x_D(t), \dot{x}_D(t), t \in [0, T] | x_D(T) = X, x_D(0) = x_0, x_D(t) - Lx_0 > 0, 0 < L < 1, 0 < x_0 < X\} \).

### 5.3 Social Planner’s Problem

We first consider the scenario that a social planner exists. The aim of the social planner is to maximize the sum of the utilities of all parties (charging station and the customers), which is called social welfare, by managing the charging rate profile over the time horizon. The optimal charging rate profile that maximizes the social welfare is called the social optimal solution. Specifically, combining the utility of the station \((5.2)\) and the customer’s utility \((5.4)\), the social welfare is defined as follows.

\[
S = \int_0^T \alpha \dot{x}_D(t) - \beta \dot{x}_D^2(t) - r(t) \dot{x}_S(t) \, dt.
\]

Note that the price term \( p(t) \) does not appear in the social welfare since it is only a utility transfer between both players. Formally, we define the social planner’s problem as follows.
Definition 7 (Social Planner’s Problem).

\[
\max_{\dot{x}_D(t), \dot{x}_S(t)} \int_0^T \alpha \dot{x}_D(t) - \beta \dot{x}_D^2(t) - r(t) \dot{x}_S(t) dt, \tag{5.5}
\]

s.t. \( \dot{x}_S(t) = \dot{x}_D(t) \), \tag{5.6}

\( x_D(t) - Lx_D(0) \geq 0 \), \tag{5.7}

\( \dot{x}_S(t) \geq 0, x_S(0) = x_D(0) = x_0, x_S(T) = x_D(T) = X \). \tag{5.8}

Note the constraint (5.7) can be omitted since the constraint \( \dot{x}_S(t) \geq 0 \) must hold, and (5.6) implies that \( x_D(t) \) is nondecreasing in the time horizon \([0, T] \) which guarantees that \( x_D(t) \geq x_D(0) \) for \( t \in [0, T] \).

Theorem 5.3.1. The socially optimal charging profile as a solution to the problem (5.5) is determined as

\[
\dot{x}^*(t) = \frac{1}{2\beta} (\alpha - r(t) - \lambda)^+. \tag{5.9}
\]

where \( \lambda \) is a constant, which can be determined by solving the water-filling problem as follows.

\[
\frac{1}{2\beta} \int_0^T (\alpha - \lambda - r(t))^+ dt = X - x_0. \tag{5.10}
\]

Proof. We construct a system with dynamics \( \dot{x}(t) = u(t) \). The problem (5.5) can be written as the following optimal control problem.

\[
\min_{u(t)} \int_0^T (r(t) - \alpha)u(t) + \beta u^2(t) dt, \tag{5.11}
\]

s.t. \( \dot{x}(t) = u(t), u(t) \geq 0, x(0) = x_0, x(T) = X. \)
The Hamiltonian of the problem (5.11) is written as

$$H(u(t), x(t), \lambda(t)) = (r(t) - \alpha)u(t) + \beta u^2(t) + \lambda u(t).$$

The state and costate equations are

$$\dot{x}^*(t) = \frac{\partial H}{\partial \lambda} = u^*(t), \quad \dot{\lambda}^*(t) = -\frac{\partial H}{\partial x} = 0.$$  \hspace{1cm} (5.12)

By (5.12), the Lagrange multiplier $\lambda(t)$ is a constant, denoted by $\lambda$. According to the minimum principle, the optimal input $u^*(t)$ must minimize the Hamiltonian as follows,

$$\min_{u(t)} H = (r(t) - \alpha)u(t) + \beta u^2(t) + \lambda u(t).$$  \hspace{1cm} (5.13)

Solving the minimization problem (5.13) and using $\dot{x}^*(t) = u^*(t)$, we obtain (5.14). The multiplier $\lambda$ can be determined by the the initial and final states, which is the solution to the following integral equation.

$$\frac{1}{2\beta} \int_0^T (\alpha - \lambda - r(t))^+ \, dt = X - x_0.$$ \hspace{1cm} (5.14)

We will later use the charging profile of the social planner’s problem to compare with the solution obtained at competitive equilibrium and Stackelberg equilibrium. Also it is an important criterion to evaluate the level of social welfare achieved at an equilibrium and compare it to the optimal social welfare.
5.3.1 Social Planner’s Problem in the Batch Arrival Case

In this section, we consider the scenario that multiple customers arrive at the station as a batch with their arrival times known to the station. In other words, during a given time interval $[0, T]$, there are $N$ PEV customers that arrive at the station at given time $0 \leq t_1 \leq t_2 \leq \ldots t_N < T$. The station knows the exact arrival times $t_1, t_2, \ldots, t_N$ and then it can generate the optimal charging rate based on the whole batch of PEV customers. This models the scenario that the PEV customers are service subscribers with charging schedules sent to station beforehand.

We study the social welfare of charging business which combines the utility of the batch of customers $i = 1, 2, \ldots, N$ and the station. In the batch arrival scenario, the customer’s problem is the same as defined in (5.4). The station’s problem is defined as follows.

\[
\max_{\dot{x}_{i,S}(t)} \int_0^T \sum_{i=1}^{N(t)} (p_i(t) - r(t)) \dot{x}_{i,S}(t) dt. \tag{5.15}
\]

\[
\text{s.t. } \sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, \tag{5.16}
\]

\[
x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \text{ for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}, \tag{5.17}
\]

**Definition 8** (Social Welfare Problem in the batch arrival case). The social welfare of the batch arrival scenario is given as

\[
S = \int_0^T \sum_{i=1}^{N(t)} (\alpha_i \dot{x}_{i,D}(t) + \beta_i \ddot{x}_{i,D}(t) - r(t) \dot{x}_{i,S}(t)) dt. \tag{5.18}
\]

Here, the function $N(t)$ is the number of the customers who have arrived at the station by time $t$, which is defined as

\[
N(t) = |\mathcal{A}(t)|, \mathcal{A}(t) = \{i = 1, 2, \ldots, N | t_i \leq t\} \tag{5.19}
\]

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In other words, the set $A(t)$ is the set of the customer who arrives at the station by time $t$ and $N(t)$ is the number of the customers who arrives by $t$. This function is given to the station since the arrivals of the customers is known a priori.

**Example 7.** We give an example of the function $N(t)$ which is related to 6 customers with arrival times $\{0, 2, 3, 4, 5, 6, 8, 7\}$. The function $N(t)$ is plotted in Figure 5.1.

![Graph of N(t)](image)

Figure 5.1. Example of the function $N(t)$ to record the arrivals of the customers.
The social planner's problem is defined as

$$\max \int_0^T \left( \sum_{i=1}^{N(t)} \left( \alpha_i \dot{x}_{i,D}(t) + \beta_i \dot{x}_{i,D}^2(t) - r(t)\dot{x}_{i,S}(t) \right) dt \right)$$

subject to

$$\dot{x}_{i,D}(t) = \dot{x}_{i,S}(t),$$

$$\sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, x_{i,D}(t) - L_i x_{i,D}(0) \geq 0, \text{for } t \in [t_i, T],$$

$$x_{i,D}(0) = x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = \dot{x}_{i,D}(t) = 0,$$

for $$t \in [0, t_i], x_{i,D}(T) = x_{i,S}(T) = X_i.$$ (5.23)

Note that the terms about the market price $$p(t)$$ does not appear in the social utility. This is because it is only a monetary transfer between two parties, which is canceled out in the sum of the utilities of the station and customers.

We characterize the charging profile at the social optimal solution to the problem (5.20) in the following result, which relies on the following definitions and notations.

- Denote the trajectory $$x^*(t), \dot{x}^*(t), t \in [0, T]$$ as the solution to the social welfare problem (5.20).
- Denote the trajectory $$u_i^c(t)$$ defined over $$t = [t_i, T]$$ as

$$u_i^c(t) = \frac{1}{2\beta_i} \left( \alpha_i - \eta_i^* - r(t) \right).$$

(5.24)

Definition 9. Define the customer set $$J(t) = \{ i \in \mathcal{A}(t) | u_i^c(t) \leq 0, x_i^*(t) - L_i x_i^*(0) > 0 \}$$. Define $$J^c$$ as the complement of $$J(t)$$: $$J^c(t) = \{ 1, 2, \ldots, N(t) \} / J(t), \text{for } t \in [0, T]$$ where $$\mathcal{A}(t)$$ is defined in (5.19).

With these definitions, we are ready to state the following result which characterizes $$x^*(t), \dot{x}^*(t).$$

Theorem 5.3.2. The social optimal charging profiles for the batch arrival scenario
are given as follows.

\[
\dot{x}_i^*(t) = \begin{cases} 
0, & \forall t \in [0, t_i), \\
u_0^i(t) & \text{if } \sum_{i=1}^{N(t)} u_0^i(t) > 0, \forall t \in [t_i, T], \\
u_i^*(t) = \frac{1}{2\beta_i} (\alpha_i - \eta_i^*) - \frac{(2\beta_i)^{-1}}{\sum_{k \in j \cap (2\beta_k)} - 1} \sum_{k \in j^c} \frac{1}{2\beta_k} (\alpha_k - \eta_k^*), & \text{if } \sum_{i=1}^{N(t)} u_i^o(t) \leq 0, x_i^*(t) - L_i x_i^*(0) > 0, \forall t \in [t_i, T], \\
0 & \text{if } x_i^*(t) - L_i x_i^*(0) = 0, \forall t \in [t_i, T], 
\end{cases}
\]

(5.25)

for \(i = 1, 2, \ldots, N(t)\), where \(N(t)\) is defined in (5.19), which is the number of the customers who have arrived at the station by time \(t\). The constant \(\eta_i^*\) can be determined by solving the boundary value problem (5.25) as unknown parameters.

Proof. Using the constraint (5.21), the problem (5.20) can be rewritten as

\[
\max_{x_i(t), \dot{x}_i(t)} \int_0^T \sum_{i=1}^{N(t)} \left( \alpha_i \dot{x}_i(t) + \beta_i \dot{x}_i^2(t) - r(t) \dot{x}_i(t) \right) dt, 
\]

s.t. \(\sum_{i=1}^{N(t)} \dot{x}_i(t) \geq 0, x_i(t) - L_i x_i^*(0) \geq 0\) 

(5.26)

(5.27)

(5.28)

Define \(N\) linear systems with dynamics given as \(\dot{x}_i(t) = u_i(t)\) for \(i = 1, 2, \ldots, N\). The
problem (5.29) is corresponding to the following optimal control problem.

\[
\min_{u_i(t)} \int_0^T N(t) \sum_{i=1}^{N(t)} \left( (r(t) - \alpha_i)u_i(t) + \beta_i u_i^2(t) \right) dt,
\]

(5.29)

s.t. \( \dot{x}_i(t) = u_i(t) \),

(5.30)

\( \sum_{i=1}^{N(t)} u_i(t) \geq 0 \),

(5.31)

\( x_i(t) - L_ix_i(0) \geq 0 \),

(5.32)

\( x_i(0) = x_{i,0}, u_i(t) = 0 \), for \( t \in [0, t_i] \), \( x_i(T) = X_i \).

(5.33)

This is an optimal control problem with a state constraint. The extended Hamiltonian for the problem is written as

\[
H = \sum_{i=1}^{N(t)} (r(t) - \alpha_i)u_i(t) + \sum_{i=1}^{N(t)} \beta_i u_i^2(t) + \sum_{i=1}^{N(t)} \eta_i(t)u_i(t) + \sum_{i=1}^{N(t)} \xi_i(t)(L_ix_i(0) - x_i(t)).
\]

where \( \mu_i(t) \) are Lagrange multipliers associated with the dynamic equation, while \( \eta_i(t) \) are Lagrange multipliers associated with the state constraint (5.32). At the optimal solution, the state and costate equations are given by

\[
\dot{x}_i^*(t) = \frac{\partial H}{\partial \mu_i^*} = u_i^*(t),
\]

(5.34)

\[
\dot{\eta}_i^*(t) = -\frac{\partial H}{\partial \mu_i^*} = \xi_i^*(t).
\]

(5.35)

By Pontryagin’s minimum principle, the Hamiltonian satisfies

\[
H(x_i^*(t), u_i^*(t), \eta_i^*(t)) \leq H(x_i^*(t), u_i(t), \eta_i^*(t))
\]

for any admissible \( u_i(t), i = 1, \ldots, N(t) \). Thus, the optimal \( u_i^*(t) \) must minimize the Hamiltonian under the constraint (5.31). For any time \( t \in [0, T] \), an optimization
problem can be written as

$$\min_{u_i(t), i=1, \ldots, N(t)} \mathcal{H}(u_i(t))$$

subject to $\sum_{i=1}^{N(t)} u_i(t) \geq 0$.

The Lagrangian of the problem is given as

$$\mathcal{L} = \sum_{i=1}^{N(t)} (r(t) - \alpha_i) u_i(t) + \sum_{i=1}^{N(t)} \beta_i u_i^2(t) + \sum_{i=1}^{N(t)} \eta_i(t) u_i(t) + \psi(t) \left( -\sum_{i=1}^{N(t)} u_i(t) \right)$$

$$+ \sum_{i=1}^{N(t)} \xi_i(t) (Lx_i(0) - x_i(t)).$$

where $\psi(t)$ is the Lagrange multiplier associated with the sum rate constraint (5.31).

With the optimal value of $u_i^*(t)$, $\eta_i^*(t)$ and $\psi^*(t)$, the Lagrangian satisfies $\frac{\partial \mathcal{L}}{\partial u_i(t)} = 0$, which implies

$$r(t) - \alpha_i + 2\beta_i u_i^*(t) + \eta_i^*(t) - \psi^*(t) = 0. \quad (5.36)$$

There are three cases possible.

**Case I:** The sum rate constraint (5.31) is not binding and (5.32) is not binding. This implies that $\psi^*(t) = 0$. The optimal input can be derived from (5.36), which is given as $\dot{x}^*(t) = u^*(t) = u_0^*(t)$.

**Case II:** The sum rate constraint (5.31) is binding and the state constraint (5.32) for any customer $i = 1, \ldots, N(t)$ is not binding: Summing the equation (5.36) over $i = 1, \ldots, N(t)$ and using $\sum_{i=1}^{N(t)} u_i^*(t) = 0$, we can obtain

$$\psi^*(t) = \frac{1}{\sum_{k=1}^{N(t)} (2\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{2\beta_k} (r(t) + \eta_k^* - \alpha_k). \quad (5.37)$$
Using (5.36) and (5.37), the optimal charging rate for customer \( i \) is given by

\[
\dot{x}_i^*(t) = u_i^*(t) = \frac{1}{2\beta_i} (\alpha_i - \eta_i^*) + \frac{(2\beta_i)^{-1}}{\sum_{k=1}^{N(t)} (2\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{2\beta_k} (\eta_k^* - \alpha_k).
\] (5.38)

**Case III:** some of the customers may have a binding state constraint (5.32) when the sum rate constraint (5.31) for all customers is binding. The set of such customers is \( J \) and other customers not in this set is in the set \( J^c \), according to Definition 9. For the customers \( i \in J^c \), the inputs must satisfy \( \sum_{i \in J^c} u_i^*(t) = 0 \), since the constraint (5.31) is binding. The optimal inputs can be obtained by summing (5.36) over \( i \in J^c \).

\[
\dot{x}^*(t) = u_i^*(t) = \frac{1}{2\beta_i} (\alpha_i - \eta_i^*) + \frac{(2\beta_i)^{-1}}{\sum_{k \in J^c} (2\beta_k)^{-1}} \sum_{k \in J^c} \frac{1}{2\beta_k} (\eta_k^* - \alpha_k).
\] (5.39)

for \( i \in J^c \). Note that in this case, the state constraint (5.32) is not binding for \( i \in J^c \), which implies \( \xi_i^*(t) = 0 \). Then, by (5.35), the parameter \( \eta_i^* \) is constant for each \( i \in J^c \).

Summing up the equation (5.36) over \( i \in J^c \), we obtain the value of \( \psi^*(t) \) in this case

\[
\psi^*(t) = \frac{1}{\sum_{i \in J^c} (2\beta_i)^{-1}} \sum_{i \in J^c} (2\beta_i)^{-1} (\eta_i^* + r(t) - \alpha_i)
\] (5.40)

For the customers in \( J \), the input must be zero; thus

\[
\dot{x}_j^*(t) = u_j^*(t) = 0, j \in J.
\] (5.41)

The Lagrange multipliers \( \eta_j^*(t), j \in J \) can be obtained using (5.36) and (5.40) by substituting \( u_j^*(t) = 0 \); thus,

\[
\eta_j^*(t) = \alpha_j + \frac{1}{\sum_{k \in J^c} (2\beta_k)^{-1}} \sum_{k \in J^c} \frac{1}{2\beta_k} (\eta_k^* - \alpha_k).
\] (5.42)
for any $j \in J$. Note that since $\eta_k^*$ is constant for each $k \in J^c$, $\eta_j^*(t)$ is also constant for $j \in J$.

Thus, for all the three cases, the optimal charging rate profile $x_i^*(t)$ and Lagrange multiplier $\eta_i^*$ are determined by solving the boundary value problem (5.25).

5.4 Competitive Equilibrium Problem

In a competitive electric vehicle charging market, there is no such a social planner to maximize the social welfare as discussed in Section 5.3. There are only market players such as charging station owner and the PEVs. Thus, it is important to analyze the interaction between the charging station owner and the PEVs. We consider a competitive market setting in which prices are *exogenous* to the players’ decisions:In order to analyze the interaction between the charging station owner and the PEVs, we further assume a competitive market setting in which prices are *exogenous* to the players’ decisions:

**Assumption 10.** The price for charging is *exogenous* in the sense that neither party has the power to manipulate the price. The price $p(t)$ is formed by a long-term evolvement of the market.

Assumption 10 is also referred as ‘pricing taking’ assumption. It excludes the case that either party influence the $p(t)$ by changing the amount of demand or supply, comparing to the classic Cournot or Bertrand games.

Therefore, the charging station cannot make its profit arbitrarily high by choosing any price profile $p(t)$ or charging profile $x(t)$. Instead, the station maximizes its profit by choosing appropriate $x_S(t)$ under the market price $p(t)$. At the same time, the PEV owner maximizes its profit by choosing appropriate $x_D(t)$ under the market price $p(t)$. An interesting problem is that whether there exists a charging profile which maximizes both player’s utilities at the same time and under the same market
price \( p(t) \). To study this problem, we introduce the concept of competitive equilibrium.

**Definition 10** (Competitive Equilibrium). A charging profile \( \dot{x}^*(t) \) such that \( \dot{x}^*(t) \in C[0, T] \) is a competitive equilibrium if and only if

- The charging profile \( \dot{x}^*(t) \) solves the charging station’s problem \((5.2)\).
- The charging profile \( \dot{x}^*(t) \) solves the PEV customer’s problem \((5.4)\).

In what follows, we present different schemes for characterizing the interaction between the PEVs and the charging station. We start analyzing the case in which a single (which might reflect an aggregate of multiple) PEV arrives to the charging station in Section \[5.4.1\]. A characterization of a competitive equilibrium is presented. The next scheme considers the arrival of a batch of PEVs at the same time, which will be presented in Section \[5.4.2\]. Finally, we focus on the problem in which PEVs arrive in a sequential way.

### 5.4.1 Single Customer and Single Station

In this part, we study the competitive equilibrium in the setting of a single PEV customer and a single station. The existence of the competitive equilibrium relies on the shape of the market price \( p(t) \) and other conditions. Since the competitive equilibrium is the optimal solution to both charging station’s problem \((5.2)\) and customer’s problem \((5.4)\), we investigate the properties of the optimal solutions to both problems and derive the expressions and properties of the competitive equilibrium and the associated market price. First, we consider the customer’s problem \((5.4)\).

The following proposition gives the expression of the optimal solution \( \dot{x}_D(t) \).

**Proposition 5.4.1.** The optimal solution \( \dot{x}_D^*(t) \) is given as

\[
\dot{x}_D^*(t) = \frac{1}{2\beta} \left( \alpha - \lambda^*(t) - p(t) \right),
\]

(5.43)
where $\lambda^*(t)$ is the Lagrange multiplier in the optimal solution to the customer's problem whose trajectory can be obtained by solving the following binary value problem

$$
\dot{x}_D^*(t) = \frac{1}{2\beta} (\alpha - \lambda^*(t) - p(t)),
$$

$$
\dot{\lambda}^*(t) = \begin{cases} 
0 & \text{if } x_D^*(t) - Lx_0 > 0, \\
-\dot{p}(t) & \text{if } x_D^*(t) - Lx_0 = 0.
\end{cases}
$$

(5.44)

for boundary condition $x_D^*(0) = x_0, x_D^*(T) = X$. The constant $x_0, L$ are given in the customer’s problem (5.4).

Proof. We prove by solving the customer’s problem (5.4).

The problem (5.4) can be formulated into a constrained optimal control problem by constructing a dynamic system $\dot{x}_D(t) = u_D(t)$ with the SoC $x_D(t)$ being a state variable and $u_D(t)$ being the system input. The system input $u_D(t)$ is the charging rate that needs to be characterized. The optimal control problem is to minimize the negative utility (5.4) over the time interval $[0, T]$. The problem is written as follows.

$$
\min_u \int_0^T \beta u_D^2(t) + (p(t) - \alpha)u_D(t)dt,
$$

$$
\text{s.t. } \dot{x}_D(t) = u_D(t), x_D(0) = x_0, x_D(T) = X, x_D(t) \geq Lx_D(0).
$$

(5.45) (5.46)

The problem (5.60) is an optimal control problem with constraint on states. Denote the set for the value of the state $x(t)$ as $\mathcal{F} = \{x \in \mathbb{R} | x - Lx_0 \geq 0\}$. The problem (5.60) can be solved by considering the augmented Hamiltonian defined on the boundary of the set $\partial \mathcal{F} = \{x \in \mathbb{R} | x - Lx_0 = 0\}$ over the time interval $t \in [s_0, s_1]$ where $s_0$ is the time instant when the state hits the boundary and $s_1$ is the time instant when the
state leaves the boundary. Specifically, the augmented Hamiltonian is given as

\[ H(x_D(t), u_D(t), \lambda(t), \eta(t)) \]
\[ = \beta u_D^2(t) + (p(t) - \alpha)u_D(t) + \lambda(t)u_D(t) + \eta(t)(Lx_D(0) - x_D(t)), \]  

where the multiplier $\lambda(t)$ is associated with the dynamic equation $\dot{x}_D(t) = u_D(t)$ and the multiplier $\eta(t)$ is associated with the state constraint $Lx_D(0) - x_D(t)$. This multiplier satisfies

\[ \eta^*(t) = 0 \text{ if } Lx_D(0) - x_D^*(t) > 0, \]  
\[ \eta^*(t) > 0 \text{ if } Lx_D(0) - x_D^*(t) = 0. \]

Note that when $Lx_D(0) - x_D^*(t) > 0$ and $\eta^*(t) = 0$, the augmented Hamiltonian becomes the classic Hamiltonian.

The necessary condition for $u_D^*(t), x_D^*(t)$ to reach the minimum of the Hamiltonian is that the following state and costate equations hold.

\[ \dot{x}^*(t) = \frac{\partial H}{\partial \lambda} = u_D^*(t), \]  
\[ \dot{\lambda}^*(t) = -\frac{\partial H}{\partial x} = -\eta^*(t). \]  

If the state $Lx_D(0) - x_D^*(t) > 0$, then $\eta^*(t) = 0$ and the multiplier $\lambda^*(t)$ becomes a constant.

By the minimum principle, the optimal input $u_D^*(t)$ must satisfy

\[ H(x_D^*(t), u_D^*(t), \lambda^*(t), \eta^*(t)) \leq H(x_D^*(t), u_D(t), \lambda^*(t), \eta^*(t)), \]  

for any admissible $u_D(t)$. Thus, for time instant $t$ such that $x_D^*(t) \notin \partial F$, the optimal input and can be obtained by minimizing the Hamiltonian \[5.47\]. Such an optimal
input can be obtained as

\[
\dot{x}_D^*(t) = u_D^*(t) = \frac{1}{2\beta} (\alpha - \lambda^*(t) - p(t)). \tag{5.53}
\]

Denote the state constraint as \( s(x_D(t), u_D(t)) \triangleq Lx_D(0) - x_D(t) \). At the time instant when the state reaches the boundary \( x_D^*(t) = Lx_0 \), the following condition must hold,

\[
\frac{d}{dt} s(x_D^*(t), u_D^*(t)) = 0.
\]

Thus, we obtain

\[
\dot{x}_D^*(t) = 0. \tag{5.54}
\]

Moreover, on the state boundary, the multiplier \( \eta^*(t) > 0 \), which implies that the multiplier \( \lambda(t) \) becomes time varying and (5.54) holds. In order to let (5.53) hold in this case, the Lagrange multiplier \( \lambda^*(t) \) is then determined as

\[
\lambda^*(t) = \begin{cases} 
\lambda^*(t^-) & \text{if } x_D^*(t) - Lx_D(0) > 0, \\
\alpha - p(t) & \text{if } x_D^*(t) - Lx_D(0) = 0.
\end{cases} \tag{5.55}
\]

where \( t^- = \sup_{\tau \leq t} \{ \tau \mid x(\tau) = Lx_0 \} \) is the last time instant that the state hits the boundary \( \partial F \) before \( t \). The multiplier \( \lambda^*(t) \) remains a constant for the time interval when the state is off the boundary \( \partial F \). Thus, if \( x \in \partial F \) at time \( t \), then \( \lambda^*(t) = \alpha - p(t) \) and the \( \dot{x}_D^*(t) = 0 \). To determine the profile \( x_D^*(t), \lambda^*(t) \), we can write the differential
equation of $\lambda^*(t)$ based on (5.56).

$$\dot{\lambda}^*(t) = \begin{cases} 0 & \text{if } x^*_D(t) - Lx_0 > 0, \\ -\dot{p}(t) & \text{if } x^*_D(t) - Lx_0 = 0. \end{cases}$$ (5.56)

Combining with the equation (5.53) and the boundary conditions $x^*_D(0) = x_0, x^*_D(T) = X$, we can obtain the profile of $\dot{x}^*_D(t)$ and $\lambda^*(t)$ as stated in the theorem by solving the boundary value problem (5.44).

**Example 8.** Suppose the market price is given as $p(t) = \sin(t) + 1.1$. The customer’s preference $\alpha = 2, \beta = .1$. We plot the optimal solution to the customer’s problem in Figure 5.2. It can be seen that when the SoC $x_D(t)$ is off the boundary $x_D(t) = Lx_0$, the Lagrange multiplier $\lambda^*(t)$ is constant. Once the SoC hits the boundary $x_D(t) = Lx_0$, the Lagrange multiplier varies with time and keeps the charging rate to be zero in order to keep SoC on the boundary.

**Remark 5.4.2.** The Proposition 5.4.1 characterizes the optimal charging profile of the customer for any given price profile $p(t)$. This is later used to characterize the charging rate profile and price profile at the competitive equilibrium.

Next, we proceed to discuss the competitive equilibrium. We restate the following classic result about a special case in minimum principle [30] in the following lemma without proof.

**Lemma 5.4.3.** Consider the optimal control problem,

$$\min_{u(t)} \int L(x(t), \dot{x}(t), u(t)) dt,$$

$$s.t. \dot{x}(t) = u(t),$$

where the state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathcal{U}$, where $\mathcal{U}$ is the feasible set for input $u(t)$.
Figure 5.2. Optimal solution to the customer’s problem
The Hamiltonian can be defined as

\[ H(x(t), \dot{x}(t), u(t), \mu(t)) = L(x(t), \dot{x}(t), u(t)) + \mu(t)u(t). \]

If the Hamiltonian \( H(u^*(t), x^*(t), \mu^*(t)) \) does not depend explicitly on the time \( t \) and the end time \( T \) is fixed, the optimal control input \( u^*(t) \) satisfies

\[ H(x^*(t), \dot{x}^*(t), u^*(t), \mu^*(t)) = c, \tag{5.57} \]

for any \( t \in [0, T] \), where \( c \) is a constant.

The following theorem characterizes the charging rate profile at the competitive equilibrium.

**Theorem 5.4.4.** Suppose that the competitive equilibrium \( (\dot{x}^*(t), p^*(t)) \) defined in Definition 10 exists. The charging profile \( \dot{x}^*(t) \) at the competitive equilibrium is represented as follows,

\[ \dot{x}^*(t) = \frac{1}{2\beta}(\alpha - r(t) - \gamma_0)^+. \tag{5.58} \]

The charging rate profile \( \dot{x}^*(t) \) over \( [0, T] \) as well as the value of \( \gamma_0 \) is obtained by solving the boundary value problem as follows with \( x(0) = x_0, x(T) = X \), where the parameter \( \gamma_0 \) is an unknown parameter in the boundary value problem.

The price profile \( p^*(t) \) at the competitive equilibrium is obtained as

\[ p^*(t) = \min\{\alpha - \lambda_0^*, r(t) + \mu_0^*\}, \tag{5.59} \]

where \( \lambda_0^* \) and \( \mu_0^* \) are any constants that satisfies \( \gamma_0 = \lambda_0^* + \mu_0^* \).

**Proof.** The outline of the proof is as follows. First, we investigate the station’s problem (5.2) under optimal control framework. Then, we prove several important
inequalities about the optimal solution \( \dot{x}_S^*(t) \) in 3 steps. Finally, we complete the proof by these inequalities.

Now, we consider the station’s problem (5.2). Similar to the problem (5.4), we reformulate the problem into an optimal control problem. We also view the SoC \( x_S(t) \) as a state variable of a dynamic system. The dynamics of the system is written as \( \dot{x}_S(t) = u_S(t) \), where \( u_S(t) \) is the system input. The optimal control problem is to minimize the negative profit (5.2) over the time interval \([0, T]\). The problem is formulated as follows.

\[
\min_u \int_0^T (r(t) - p(t))u_S(t)dt, \quad (5.60)
\]

s.t. \( \dot{x}_S(t) = u_S(t), u_S(t) \geq 0, x_S(0) = x_0, x_S(T) = X. \quad (5.61)\]

To solve this problem, we consider the Hamiltonian.

\[
\mathcal{H}(u_S(t), x_S(t), \mu_0(t)) = (r(t) - p(t) + \mu_0(t))u_S(t), \quad (5.62)
\]

where \( \mu_0(t) \) is the Lagrangian multiplier. At the optimal solution, the following the state and costate equations hold

\[
\dot{x}_S^*(t) = \frac{\partial \mathcal{H}}{\partial \mu_0^*} = u_S^*(t), \quad \dot{\mu}_0^*(t) = -\frac{\partial \mathcal{H}}{\partial x_S^*} = 0. \quad (5.63)
\]

Thus, the Lagrange multiplier \( \mu_0^*(t) \) is a constant, which is denoted as \( \mu_0^*. \) \( i.e. \)

\[
\mu_0^*(t) = \mu_0^*. \quad (5.64)
\]

Next, we prove several inequalities of optimal charging profile \( \dot{x}_S^*(t) \) in terms of \( \mu_0^*, r(t), p(t) \) by following the next 3 steps. The charging rate at the equilibrium is obtained in Step 4 using the results proved in steps 1-3. Then, in the Step 5, we ver-
ify that this solution, which is derived from necessary conditions, actually maximizes both utility functions of the customer and the station.

**Step 1:** We show that the optimal solution of the station’s problem \((5.2)\) satisfies

\[
\dot{x}^*_S(t) = 0, \quad \text{if} \quad r(t) - p(t) + \mu^*_0 > 0. \tag{5.65}
\]

for any \(t \in [0, T]\), where \(\mu^*_0\) is a constant Lagrange multiplier defined in \(5.64\).

The statement \((5.65)\) can be shown as follows. By the minimum principle, the optimal input \(u^*_S(t)\) satisfies

\[
\mathcal{H}(u^*_S(t), x^*_S(t), \mu^*_0(t)) \leq \mathcal{H}(u_S(t), x^*_S(t), \mu^*_0(t)), \tag{5.66}
\]

for any admissible input \(u_S(t)\) at any \(t \in [0, T]\). Thus, the optimal input \(u^*_S(t)\) is the solution to the following problem.

\[
\min_{u_S(t)} (r(t) - p(t) + \mu^*_0)u_S(t), \tag{5.67}
\]

\[
s.t. \quad u_S(t) \geq 0, \quad \int_0^T u_S(t)dt = X - x_0. \tag{5.68}
\]

Since the admissible input \(u_S(t) \geq 0\), at the time \(t\) when \(r(t) - p(t) + \mu^*_0 > 0\), the solution \(u^*_S(t) = 0\) minimizes the Hamiltonian \((5.62)\). Thus, \(\dot{x}^*_S(t) = u^*_S(t) = 0\). Based on the minimum principle, the following result holds for the optimal input \(u^*_S(t)\).

**Step 2:** We show that at the optimal solution of the station’s problem \((5.2)\), the following condition holds for any \(t \in [0, T]\),

\[
r(t) - p(t) + \mu^*_0 \geq 0 \tag{5.69}
\]

where \(\mu^*_0\) is the constant Lagrange multiplier defined in \((5.64)\). Moreover, the constant
$c$ defined in Lemma 5.4.3 satisfies $c = 0$.

The inequality (5.69) can be shown by by deriving contradictions to exclude the following two cases.

(i) There exists some $t \in [0, T]$, such that $r(t) - p(t) + \mu^*_0 < 0$ holds and also some $t \in [0, T]$, such that $r(t) - p(t) + \mu^*_0 \geq 0$

(ii) For any $t \in [0, T]$, $r(t) - p(t) + \mu^*_0 < 0$.

Suppose the case (i) holds. Consider the time instant $t$ such that $r(t) - p(t) + \mu^*_0 < 0$ holds. According to the minimum principle, the optimal input $u^*_S(t)$ should minimize the Hamiltonian at this time $t$. Note that the input $u^*_S(t) = 0$ cannot achieve the minimum of Hamiltonian, because there exists an admissible input with $u^*_S(t) = \epsilon > 0$ such that the Hamiltonian $\mathcal{H}(x^*(t), \epsilon, \mu^*_0) < \mathcal{H}(x^*(t), 0, \mu^*_0)$. Recall that the admissible input should satisfy $u^*_S(t) \geq 0$, thus, the optimal $u^*_S(t) > 0$.

According to the condition (5.57), the optimal input at $t$ satisfies

$$u^*(t) = \frac{c}{r(t) - p(t) + \mu^*_0}. \quad (5.70)$$

where the constant $c < 0$ because $u^*(t) > 0$ and $r(t) - p(t) + \mu^*_0 < 0$ at the time $t$.

In the case (i), consider the time instant $t'$ such that $r(t') - p(t') + \mu^*_0 \geq 0$. If $r(t') - p(t') + \mu^*_0 = 0$, then (5.57) indicates that the constant $c = 0$. If $r(t') - p(t') + \mu^*_0 > 0$, then Step 1 implies that $u^*(t') = 0$ and the Hamiltonian at the time $t'$ is equal to zero, which implies that $c = 0$. In summary, the constant $c = 0$, if $r(t') - p(t') + \mu^*_0 \geq 0$.

But at time $t$ such that $r(t) - p(t) + \mu^*_0 < 0$, the constant $c < 0$. Since this constant $c$ should be the same value over the time interval $[0, T]$, this leads to the contradiction. Thus, the case (i) can never exist.

Next, we consider the case (ii). According to the condition (5.57), the optimal
input over the entire horizon \( t \in [0, T] \) satisfies

\[
u^*_S(t) = \frac{c}{r(t) - p(t) + \mu_0^*}. \tag{5.71}
\]

According to the minimum principle, this input \( u^*_S(t) \) minimizes the Hamiltonian, which satisfies (5.66), i.e.

\[
\mathcal{H}(x^*_S(t), u^*_S(t), \mu_0^*) = c \leq (r(t) - p(t) + \mu_0^*)u_S(t), \tag{5.72}
\]

for any admissible input \( u_S(t) \). Note that the input \( u_S(t) \) only need to satisfy the integral equality in (5.68) and there is no upper bound imposed on the magnitude \(|u_S(t)|\). Given that value \(|c| > M\), for \( M > 0 \) is a arbitrary large positive number, there always exists an admissible input \( u_S(t) \) with \(|u_S(t)|\) large enough at some \( t \) such that \((r(t) - p(t) + \mu_0^*)u_S(t) < c\), which means that \( \mathcal{H}(x^*(t), u^*(t), \mu_0^*) \) is not minimum at \( t \). This leads to a contradiction. Thus, the case (ii) never holds. Since we exclude the case (i) and (ii), the only case that holds is that \( r(t) - p(t) + \mu_0^* \geq 0 \) for any \( t \in [0, T] \).

Finally, we show that \( c = 0 \). By (5.65) proved in Step 1, at the time instant \( t \) when \( r(t) - p(t) + \mu_0^* > 0 \), the optimal input satisfies \( u^*_S(t) = 0 \). Thus, the Hamiltonian \( \mathcal{H}(x^*_S(t), u^*_S(t), \mu_0^*) = 0 \) according to Lemma 5.4.3. Obviously, for any \( t \in [0, T] \) when \( r(t) - p(t) + \mu_0^* = 0 \), then \( \mathcal{H}(x^*_S(t), u^*_S(t), \mu_0^*) = 0 \). Since \( r(t) - p(t) + \mu_0^* \geq 0 \) for any \( t \in [0, T] \), then the constant \( c = 0 \).

**Step 3.** We show that the optimal input \( \dot{x}^*_S(t) \) of the station’s problem (5.2) satisfies the following conditions.

\[
r(t) - p(t) + \mu_0^* = 0, \quad \text{if } \dot{x}^*_S(t) > 0. \tag{5.73}
\]

where \( \mu_0^* \) is a constant value such that the Lagrange multiplier at the optimal solution
\( \mu^*(t) = \mu_0^* \), as defined in (5.63), (5.64).

The inequality (5.73) can be shown as follows. By the inequality (5.69) in Step 2, we know that the constant in (5.57) satisfied \( c = 0 \). According to Lemma 5.4.3, the optimal input satisfies

\[
(r(t) - p(t) + \mu_0^*) u_S^*(t) = 0.
\]  

(5.74)

In order to satisfy the condition above, at any time \( t \in [0, T] \), if \( u_S^*(t) > 0 \), then \( r(t) - p(t) + \mu_0^* = 0 \). Also, by noting that \( u_S^*(t) = \dot{x}_S^*(t) \), the condition (5.73) holds.

**Step 4.** Now we are ready to prove the theorem. The competitive equilibrium implies that \( \dot{x}^*(t) = \dot{x}_D^*(t) = \dot{x}_S^*(t) \). Thus, both necessary conditions for the optimal solution of the customer and station’s problems must be satisfied, i.e. \( (x^*(t), \dot{x}^*(t)) \in \mathcal{X}_S \cap \mathcal{X}_D \). The equilibrium must match the optimal solution (5.43) to the customer’s problem, i.e. \( \dot{x}^*(t) = \dot{x}_D^*(t) \). Thus, \( \dot{x}^*(t) = 0 \) if \( x^*(t) = Lx(0) < x(0) \). Since \( (x^*(t), \dot{x}^*(t)) \in \mathcal{X}_D \), we know that \( \dot{x}^*(t) \geq 0 \). The state can only be greater or equal to its initial value, so \( x^*(t) \geq Lx(0) \) for any \( t \in [0, T] \). According to (5.49), the Lagrange multiplier \( \lambda^*(t) \) remains constant, i.e.

\[
\lambda^*(t) = \lambda_0^*.
\]  

(5.75)

Based on Proposition 5.4.1 and \( \dot{x}^*(t) = \dot{x}_D^*(t) \), the charging rate at the equilibrium can be represented as

\[
\dot{x}^*(t) = \frac{1}{2\beta} (\alpha - \lambda_0^* - p(t)).
\]  

(5.76)

The charging rate also satisfies \( \dot{x}^*(t) \geq 0 \), which implies that

\[
p(t) \leq \alpha - \lambda_0^*.
\]  

(5.77)
At the time $t$ when $p(t) < \alpha - \lambda^*_0$, the charging rate at the equilibrium $\dot{x}^*(t) > 0$. According to (5.73) in Step 3, we obtain $r(t) + \mu^*_0 - p(t) = 0$. In summary, for any $t \in [0, T]$, the following conditions hold,

\begin{align*}
p(t) &\leq \alpha - \lambda^*_0, \\
p(t) &\geq r(t) + \mu^*_0, \quad \text{if} \quad p(t) < \alpha - \lambda^*_0.
\end{align*}

(5.78)  (5.79)

By (5.78), it can be seen that the market price at the equilibrium falls into only two mutually exclusive scenarios. One scenario is that $p(t) < \alpha - \lambda^*_0$ for some $t \in [0, T]$, in which case $p(t) = r(t) + \mu^*_0$, which implies $r(t) + \mu^*_0 < \alpha - \lambda^*_0$. The other scenario is that $p(t) = \alpha - \lambda^*_0$. Thus, if $r(t) + \mu^*_0 \geq \alpha - \lambda^*_0$, the first case does not hold. Due to the mutual exclusiveness of the two scenarios, the second case must hold, which means that $p(t) = \alpha - \lambda^*_0$.

Moreover, we argue that if $r(t) + \mu^*_0 < \alpha - \lambda^*_0$, then $p(t) = r(t) + \mu^*_0$. This can be seen by noting that the price $p(t)$ satisfies (5.69), $p(t) \leq r(t) + \mu^*_0$, which implies that $p(t) < \alpha - \lambda^*_0$. According to (5.79), we obtain $p(t) = r(t) + \mu^*_0$.

Summarizing the results stated above, we obtain the condition (5.59).

We can find the charging rate at the competitive equilibrium by substituting (5.59) into the (5.76). The charging rate can be written as,

\[ \dot{x}^*(t) = \begin{cases} 0 & \text{if} \quad \alpha - \lambda^*_0 < r(t) + \mu^*_0, \\
\frac{1}{2\beta} (\alpha - r(t) - \lambda^*_0 - \mu^*_0) & \text{otherwise} \end{cases} \]  

(5.80)

Denote $\gamma_0 = \lambda^*_0 + \mu^*_0$, we obtain (5.58).

**Step 5.** In this step, we verify the obtained charging rate maximizes the utility functions of the customer and station’s problems.

Since we obtain the equilibrium charging rate based on the optimal solution $\dot{x}^*_D(t)$ of the customer’s problem, it remains to verify that the obtained trajectory of charg-
ing rate \( \dot{x}^*(t) \) maximizes the utility of the station’s problem. Consider the Hamiltonian (5.62) of the station’s problem.

\[
\mathcal{H}(u_S(t), x_S(t), \mu_0(t)) = (r(t) - p(t) + \mu_0(t))u_S(t),
\]

(5.81)

When the price at the equilibrium \( p(t) = r(t) + \mu_0^* \), \( \mathcal{H}(u_S(t), x_S(t), \mu_0(t)) = 0 \), the charging rate can take any value at this time. So the obtained \( \dot{x}^*(t) \) is naturally an optimal solution at such a time. When \( p(t) = \alpha - \lambda_0^* \) which implies \( \alpha - \lambda_0^* < r(t) + \mu_0^* \), thus, \( (r(t) - p(t) + \mu_0(t)) > 0 \). To minimize the Hamiltonian, \( u_S^*(t) = 0 \) should hold.

And in this case, we have \( x^*(t) = 0 \). According to the minimum principle, the solution \( x^*(t) \) minimizes the Hamiltonian (5.62) at any time. Thus, it is verified that \( u^*(t) = u_S(t) \)

\[ \square \]

Remark 5.4.5. Note that the market price at the equilibrium \( p(t) = \min\{\alpha - \lambda_0^*, r(t) + \mu_0^*\} \) is not unique. This is because the Lagrange multiplier \( \lambda_0^*, \mu_0^* \) can take any value such that \( \lambda_0^* + \mu_0^* = \gamma_0 \), with \( \gamma_0 \) being determined by the boundary value problem defined in (5.58). Thus, the equilibrium market price is a class of curves defined over \([0, T]\).

Example 9. Consider the charging business between one station and a single customer. Suppose the wholesale electricity price \( r(t) = \sin(t) + 1.1 \). The customer’s preference \( \alpha = 2, \beta = .1 \). We obtain the equilibrium charging rate by solving (5.58) and plot the charging rate and SoC at the competitive equilibrium in Figure 5.3. In this example, the Lagrange multiplier \( \gamma_0 = \lambda_0^* + \mu_0^* = 0.9546 \).

Note that the charging rate at the equilibrium becomes zero when the wholesale price is relatively high.

We select several market prices at the equilibrium by choosing multiple Lagrange multipliers from \( \lambda_0^* = 0.2 \) to \( \lambda_0^* = 0.45 \) and \( \mu_0^* \) accordingly. The profiles of the market prices, wholesale price and the equilibrium charging rate are plot in Figure 5.4.
Figure 5.3. Charging rate and SoC at the competitive equilibrium
Figure 5.4. Charging rate and SoC at the competitive equilibrium
Proposition 5.4.6. The charging profile (5.58) at the competitive equilibrium is efficient.

Proof. To obtain the charging profile (5.58) at the competitive equilibrium is equivalent to solve the water filling problem (5.14). By Theorem 5.3.1, this solution is socially optimal. Thus, the charging profile (5.58) is efficient. □

This result is consistent with Theorem 1 in [76], which indicates that the charging profiles achieved in the competitive equilibrium maximizes the social welfare.

5.4.2 The Competitive Equilibrium of the Case with Multiple Customers in a Batch Arrival

In this section, we consider the scenario that multiple customers arrive at the station as a batch with their arrival times known to the station. In other words, during a given time interval $[0, T]$, there are $N$ PEV customers that arrive at the station at given time $0 \leq t_1 \leq t_2 \leq \ldots t_N < T$. The station knows the exact arrival times $t_1, t_2, \ldots, t_N$ and then it can generate the optimal charging rate based on the whole batch of PEV customers. In this model, the following assumption is made.

Assumption 11. The station does not sell power to the grid, which implies that the sum of charging rates should be ensured to be non-negative at any time.

Definition 11 (Customer’s problem in the batch arrival case). The problem for each customer $i$, is to maximize the customer’s utility over the time interval $[t_i, T], 0 < t_1 < \ldots < t_{i-1} < t_i < t_{i+1} < \ldots < T$.

\[
\max_{\hat{x}_{i,D}(0)} \int_{t_i}^{T} (\alpha_i - p_i(t))\hat{x}_{i,D}(t) - \beta_i x^2_{i,D}(t)dt, \tag{5.82}
\]

s.t. \[x_{i,D}(0) = x_{i,0}, x_{i,D}(T) = X, \tag{5.83}\]

\[x_{i,D}(t) \geq Lx_{i,D}(0). \tag{5.84}\]
For convenience, each customer $i$’s problem is also studied under the same time frame $[0, T]$ and also extend the domain of charging profile $x_{i,D}(t)$ to be $[0, T]$ accordingly.

$$\max_{\dot{x}_{i,D}(t)} \int_0^T (\alpha_i - p_i(t))\dot{x}_{i,D}(t) - \beta_i\dot{x}_{i,D}^2(t)dt, \quad \quad \quad (5.85)$$

s.t. $x_{i,D}(t) = x_{i,0}, \dot{x}_{i,D}(t) = 0$ for $t \in [0, t_i]$, $x_{i,D}(T) = X, x_{i,D}(t) \geq Lx_{i,D}(0)$.

**Definition 12** (Station’s problem in the batch arrival). The station’s problem can be written as

$$\max_{\dot{x}_{i,S}(t)} \int_0^T N(t) \sum_{i=1}^{N(t)} (p_i(t) - r(t)) \dot{x}_{i,S}(t)dt, \quad \quad \quad (5.86)$$

s.t. $\sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, \quad \quad \quad (5.87)$

$$x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \text{for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}, \quad \quad \quad (5.88)$$

where $N(t) = |\{i = 1, 2, \ldots, N|t_i \leq t\}|$ is the number of the customers who have arrived at the station by the time $t$, which is defined in (5.19). The constraint (5.87) is imposed because of the Assumption 11.

**Definition 13** (Competitive equilibrium with multiple customers in a batch). The competitive equilibrium for the station’s problem (5.86) and the customers’ problems (5.82) in a batch arrival case is defined as a set of charging profiles $\{\dot{x}_{i}^*(t)\}_{i=1}^{N}$ defined over $[0, T]$ such that

$$\dot{x}_{i}^*(t) = \dot{x}_{i,D}^*(t) = \dot{x}_{i,S}^*(t), \quad t \in [0, T], \quad \quad \quad (5.89)$$

for each $i = 1, \ldots, N$, where $\dot{x}_{i,D}^*(t)$ is the optimal solution to the customer’s problem (5.82) and $\dot{x}_{i,S}^*(t)$ is the optimal solution to the station’s problem (5.86).
Next, we characterize the charging profile at the competitive equilibrium defined in (5.89) in the following result, which relies on the following definitions and notations.

- Denote the trajectory $x^{(c)}(t), \dot{x}^{(c)}(t), t \in [0, T]$ as the solution to the social welfare problem (5.20).
- Denote the trajectory $u^{(c)}_i(t)$ defined over $t = [t_i, T]$ as
  \[ u^{(c)}_i(t) = \frac{1}{2\beta_i} (\alpha_i - \gamma^*_i - r(t)) \]  
  (5.90)
  where $\gamma^*_i$ is a constant parameter to be determined.

Similar to Definition 9, we define the following customer set.

**Definition 14.** Define the customer set $J_c(t) = \{ i \in A(t) | x^{(c)}_i(t) - L_i x_i(0) = 0 \}$. Define $J_c^c(t)$ as the complement of $J_c(t): J_c^c(t) = \{ 1, 2, \ldots, N(t) \} / J_c(t)$, for $t \in [0, T]$ where $A(t)$ is defined in (5.19).

**Theorem 5.4.7.** The charging rate profile $\dot{x}^{(c)}_i(t)$ for the customer $i$ at the competitive equilibrium in the batch arrival setting is characterized as

\[
\dot{x}^{(c)}_i(t) = \begin{cases} 
0 & \text{if } t \in [0, t_i), \\
u^{(c)}_i(t) & \text{if } \sum_{i=1}^{N(t)} u^{(c)}_i(t) > 0, x^{(c)}_i(t) - L_i x_{i,0} > 0, t \in (t_i, T], \\
\frac{1}{2\beta_i} \left( \alpha_i - \gamma^*_i - \frac{\sum_{k \in J_c(2\beta_k)^{-1}(\alpha_k - \gamma^*_k)}}{\sum_{k \in J_c(2\beta_k)^{-1}}} \right) & \text{if } \sum_{i=1}^{N(t)} u^{(c)}_i(t) \leq 0, x^{(c)}_i(t) - L_i x_{i,0} > 0, t \in (t_i, T], \\
0 & \text{if } x^{(c)}_i(t) - L_i x_{i}(0) = 0, t \in (t_i, T],
\end{cases}
\]  
(5.91)

for $i = 1, 2, \ldots, N(t)$. where the parameter $\gamma^*_i = \mu^*_i + \lambda^*_i$, where The constants $\mu^*_i, i = 1, 2, \ldots, N$ are Lagrange multipliers in the solution of the station’s problem (5.86), and the curves $\lambda^*_i, i = 1, 2, \ldots, N$ are Lagrange multipliers in the solution of the customers’ problem (5.82). The boundary conditions for the differential equation (5.91) are $x^{(c)}_i(0) = x_{i,0}, x^{(c)}_{i}(T) = X_i$ for $i = 1, \ldots, N(t)$. 

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The parameter $\gamma^*_i$ is determined by solving the differential equation (5.91) as the unknown parameter in the boundary value problem.

At the equilibrium, the market price $p^{(c)}_i(t)$ for each customer $i$ is characterized as

\[
p^{(c)}_i(t) = \begin{cases} 
  r(t) + \mu_i & \text{if } \sum_{k=1}^{N(t)} u^{(c)}_k(t) > 0, x^{(c)}_i(t) - L_i x_{i,0} > 0, \\
  \frac{\sum_{k \in J} (\alpha_k - \lambda^{*}_k - \mu^*_k)}{\sum_{k \in J} (2\beta_k)^{-1}} + \mu^*_i, & \text{if } \sum_{k=1}^{N(t)} u^{(c)}_k(t) = 0, x^{(c)}_i(t) - L_i x_{i,0} > 0, \\
  \alpha_i - \lambda^*_i & \text{if } x^{(c)}_i(t) - L_i x_{i,0} = 0, i \in J.
\end{cases}
\] (5.92)

Proof. We first consider the station’s problem and recast this problem into an optimal control problem with system dynamics defined as $\dot{x}_S(t) = u_S(t)$. The optimal control problem is written as

\[
\max_{u_i,S} \int_0^T \sum_{i=1}^{N(t)} (p_i(t) - r(t)) u_{i,S}(t) dt.
\] (5.93)

s.t. \[ \sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, \dot{x}_{i,S}(t) = u_{i,S}(t). \] (5.94)

\[ x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \text{for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}. \] (5.95)

The Hamiltonian of this problem (5.93) is written as

\[
\mathcal{H} = \sum_{i=1}^{N(t)} (r(t) - p_i(t)) \dot{x}_{i,S}(t) + \sum_{i=1}^{N(t)} \mu_i(t) u_{i,S}(t).
\] (5.96)

Based on the minimum principle, the following state equation and costate equation hold,

\[
\dot{x}^*_i(t) = \frac{\partial \mathcal{H}}{\partial \mu_i} = u^*_i(t),
\]

\[
\dot{\mu}_i^*(t) = -\frac{\partial \mathcal{H}}{\partial x^*_i} = 0.
\]
Thus, the multipliers \( \mu_i^*(t), i = 1, 2, \ldots, N \) are constants. The optimal input \( u_i^*(t) \) minimizes the Hamiltonian for any \( t \in [t_i, T] \), which can be found through the following problem,

\[
\min_{u_i(t)} \mathcal{H} = \sum_{i=1}^{N(t)} (r(t) - p_i(t)) \dot{x}_{i,S}(t) + \sum_{i=1}^{N(t)} \mu_i(t) u_{i,S}(t) \\
\text{s.t. } \sum_{i=1}^{N(t)} u_i(t) \geq 0. \tag{5.97}
\]

This is a classic constrained optimization problem. The Lagrangian of this optimization problem is written as

\[
\mathcal{L} = \sum_{i=1}^{N(t)} (r(t) - p_i(t)) u_{i,S}(t) + \sum_{i=1}^{N(t)} \mu_i(t) u_{i,S}(t) + \eta(t) \left( -\sum_{i=1}^{N(t)} u_{i,S}(t) \right).
\]

where \( \eta(t) \) is the Lagrange multiplier associated with the constraint \( (5.97) \). At the optimal solution, the Lagrangian satisfies \( \frac{\partial \mathcal{L}}{\partial u_i(t)} = 0 \), which implies

\[
r(t) - p_i(t) + \mu_i^* - \eta^*(t) = 0. \tag{5.98}
\]

Using \( (5.98) \), we study the optimal solution based on whether the constraint \( (5.87) \) on the sum of the charging rate profile of the station and the state constraint \( (5.84) \) from the customer’s problem are binding or not. There are various cases of combinations of the two conditions and we discuss them as follows.

**Case I:** neither of the constraint \( (5.84) \) and \( (5.87) \) is binding for any customer \( i = 1, 2, \ldots, N(t) \). In this case, \( \eta^*(t) = 0 \) because the constraint \( (5.87) \) is not binding. Using \( (5.98) \), the profile of \( p^*_i(t) \) is given by

\[
p^*_i(t) = r(t) + \mu_i^*. \tag{5.99}
\]
By definition, the charging profile at the competitive equilibrium \( \dot{x}_i^{(c)}(t) = \dot{x}_{i,D}^*(t) \), which is the solution to the customer’s problem given by \( (5.43) \). Substitute \( (5.99) \) into \( \dot{x}_{i,D}^*(t) \), the equilibrium charging rate \( \dot{x}_i^{(c)}(t) \) can be obtained by \( (5.98) \), which is obtained as \( \dot{x}_i^{(c)}(t) = u_i^{(c)}(t) \), where \( u_i^{(c)}(t) \) is defined in \( (5.177) \).

Since the constraint \( (5.84) \) is not binding, the multiplier \( \lambda^*(t) \) in this case is constant. Also recall that \( \mu^*(t) \) is constant for all cases. This implies that the parameter \( \gamma(t) = \lambda_i^*(t) + \mu_i^*(t) \) is also a constant.

**Case II:** The state constraint \( (5.84) \) are not binding for any customer \( i \), but the sum constraint \( (5.87) \) is binding. Since constraint \( (5.87) \) is binding, the multiplier \( \eta^*(t) \) is nonzero, which is obtained by summing up \( (5.98) \) with weight \( 1/(2\beta_i) \),

\[
\sum_{i=1}^{N(t)} \frac{1}{2\beta_i} (r(t) - p_i(t) + \mu_i^* - \eta(t)) = 0. \tag{5.100}
\]

Thus, the Lagrange multiplier \( \eta^*(t) \) is obtained as

\[
\eta^*(t) = \frac{1}{\sum_{i=1}^{N(t)} (2\beta_i)^{-1}} \sum_{k=1}^{N(t)} (2\beta_i)^{-1} (r(t) - p_i(t) + \mu_i^*). \tag{5.101}
\]

Using \( (5.98) \) and \( (5.101) \), we obtain the price profile in this case as

\[
p_i(t) = r(t) + \mu_i^* + \frac{1}{\sum_{i=1}^{N(t)} (2\beta_i)^{-1}} \sum_{k=1}^{N(t)} (2\beta_i)^{-1} (p_i(t) - r(t) - \mu_i^*). \tag{5.102}
\]

Note that the Lagrange multiplier \( \lambda_i^*(t) \) is a constant since the constraint on the state is not binding in this case. By definition, since the sum rate constraint \( (5.87) \) is binding we also have \( \sum_{i=1}^{N(t)} \dot{x}_D(t) = 0 \), which implies

\[
\sum_{i=1}^{N(t)} \frac{1}{2\beta_i} (\alpha_i - p_i(t) - \lambda_i^*) = 0 \tag{5.103}
\]
By (5.102) and (5.103), the price profile can be obtained as
\[ p^*_i(t) = \frac{\sum_{k=1}^{N(t)}(2\beta_k)^{-1}(\alpha_k - \lambda_k^* - \mu_k^*)}{\sum_{k=1}^{N(t)}(2\beta_k)^{-1}} + \mu_i^*. \quad (5.104) \]
Substitute the price profile into \( \dot{x}^*_i(t) \), the competitive equilibrium charging rate profile in this case is
\[ \dot{x}^*_i(t) = \frac{1}{2\beta_i} \left( \alpha_i - \gamma_i^* - \frac{\sum_{k=1}^{N(t)}(2\beta_k)^{-1}(\alpha_k - \gamma_k^*)}{\sum_{k=1}^{N(t)}(2\beta_k)^{-1}} \right). \quad (5.105) \]
In this case, none of the state constraint (5.84) is binding, so \( \lambda_i \) for every \( i \) is constant. Thus, \( \gamma_i^* = \mu_i^* + \lambda_i^* \) for every \( i \).

**Case III:** The state constraint (5.84) are binding for at least one customer \( i \), while the sum constraint (5.87) is also binding for all customers. Recall that the set \( J_c \) is the set of customers with state constraint (5.84) binding. For customers in \( J_c \), the input must be zero to meet the state constraint, which is imposed by the solution to customer’s problem. We have
\[ u^*_j(t) = 0, j \in J_c. \quad (5.106) \]
since the constraint (5.87) is also binding, for the customers \( i \in J_c \), the inputs must satisfy \( \sum_{i \in J_c} u^*_i(t) = 0 \). Following the same procedure as in case II, we obtain the charging rates of the customers \( i \in J_c \) as,
\[ \dot{x}^{(c)}_i(t) = \frac{1}{2\beta_i} \left( \alpha_i - \mu_i^* - \lambda_i^* - \frac{\sum_{k \in J_c}(2\beta_k)^{-1}(\alpha_k - \lambda_k^* - \mu_k^*)}{\sum_{k \in J_c}(2\beta_k)^{-1}} \right). \]
The price profile for customer \( j \in J_c \) is
\[ p^{(c)}_j(t) = \frac{\sum_{k \in J_c}(2\beta_k)^{-1}(\alpha_k - \lambda_k^* - \mu_k^*)}{\sum_{k \in J_c}(2\beta_k)^{-1}} + \mu_i^*. \quad (5.107) \]
For customers \( j \in J_c \), at the competitive equilibrium, the price profile \( p_i^{(c)}(t) \) must take appropriate forms such that

\[
\dot{x}_{i,D}^*(t) = \frac{1}{2\beta_i} (\alpha_i - \lambda^*(t) - p_i(t)) = 0.
\]

With the last equation, the profile of price can be determined as

\[
p_i^{(c)}(t) = \alpha_i - \lambda_i^*(t). \tag{5.108}
\]

Next, we characterize \( \gamma_j^*(t) \) in this case. First consider the customers \( j \in J_c \), since the state constraint (5.84) is not binding for each customer \( j \in J_c \), then \( \lambda_j^*(t) \) is constant due to Proposition 5.4.1. Thus, \( \gamma_j^*(t) \) is constant for every \( j \in J_c \).

By (5.98) and (5.107), we obtain \( \eta^*(t) \) as

\[
\eta^*(t) = r(t) + \frac{\sum_{k \in J_c} (2\beta_k)^{-1} (\alpha_k - \lambda_k^* - \mu_k^*)}{\sum_{k \in J_c} (2\beta_k)^{-1}}. \tag{5.109}
\]

For customer \( i \in J_c \), using (5.98), (5.109) and (5.108), we obtain

\[
\gamma_i^* = \alpha_i + \frac{\sum_{k \in J_c} (2\beta_k)^{-1} (\alpha_k - \gamma_k^*)}{\sum_{k \in J_c} (2\beta_k)^{-1}}. \tag{5.110}
\]

Thus, the parameter \( \gamma_i^* \) for customer \( i \in J_c \) is also constant, which depends on \( \gamma_j^*, j \in J_c \).

**Case IV:** for every customer \( i \), the sum constraint (5.87) is not binding while the state constraint (5.84) is binding. In fact, this case actually never happens because as long as the state constraint (5.84) is binding, the charging rate \( \dot{x}_i^*(t) = 0 \), which also implies that the sum of the charging rates is also equal to zero and the sum constraint (5.87) is binding.

Summarizing all cases discussed above, we can obtain the charging rate (5.91).
and the price profile at the competitive equilibrium (5.92).

Example 10. Consider the scenario of three customers who arrive at the station to charge the vehicles at \( t_{1,0} = .1, t_{2,0} = .5, t_{3,0} = 1.6 \), respectively. The customers’ parameters are \( \alpha_i = 1, \beta = .1, i = 1, 2, 3 \). The initial values of the SoC of both vehicle are \( x_{1,0} = 0.1, x_{2,0} = .5, x_{3,0} = 1.6 \). The final SoC levels are \( X_1 = 15, X_2 = 13.5, X_3 = 10.5 \). Here, the wholesale electricity price profile is given as \( r(t) = \sin(5t) + 1.1, t \in [0, T] \).

The charging rate profiles of the three vehicles and the sum of the charging rates \( \sum_{i=1}^{3} \dot{x}_i^{(g)}(t) \) are illustrated in Figure 5.5 and Figure 5.6. We can observe that the sum of the charging rate profiles at the competitive equilibrium remains nonnegative over the entire time horizon.

The price profile at competitive equilibrium illustrated in Figure 5.7.

Proposition 5.4.8. In the batch arrival case, the charging profile (5.91) at the competitive equilibrium is efficient.

Proof. It is easy to check that the charging profile (5.91) coincides with the charging profile (5.25) which is the social optimal in the batch arrival case. Therefore, the charging profile (5.91) at the competitive equilibrium is efficient.

5.5 Stackelberg Market Equilibrium Problem

Section 5.4 discusses the competitive equilibrium under the standard information structure, in which each player does not have access to each other’s utility function and strategy. The only information shared by both players is the market price signal \( p(t) \). However, in most cases, the charging station as a corporation might have information about preferences or car types of the customers. Thus, the charging station has access to the customer’s model and adjust his strategy to achieve particular goals.
Figure 5.5. Charging rate profiles at competitive equilibrium: batch arrival

Figure 5.6. Sum of charging rate profiles at competitive equilibrium: batch arrival
Figure 5.7. Price profiles at competitive equilibrium: batch arrival

In the traditional game theory framework, such game is often called Stackelberg game [18, 27], in which the players are modeled as the leader and the follower. The leader knows the utility and strategy of the follower and makes his decision based on such information. The follower makes his decision after the moves of the leader. Section 5.4 discusses the competitive equilibrium under the standard information structure, in which each player does not have access to each other’s utility function and strategy. The only information shared by both players is the market price signal \( p(t) \). However, in most cases, the charging station as a corporation can use the professional experience and business domain knowledge to anticipate the customer’s behavior and their utility function. Thus, the charging station has access to the customer’s model and adjust his strategy to achieve particular goals. In the traditional game theory framework, such biased game is often called Stackelberg game, in which the players
are modeled as the leader and the follower. The leader knows the utility and strategy of the follower and makes his decision based on such information. The follower makes his decision after the moves of the leader.

In this section, we consider the charging business model in the Stackelberg sense.

**Assumption 12.** The station has knowledge of the customer’s utility and strategy. In other words, the station knows the optimal solution to the customer’s problem and uses it to maximize the station’s utility. Moreover, the station can determine the price \( p(t) \).

We define the customer’s problem, which is to maximize the utility under the price \( p(t) \).

**Definition 15** (Customer’s problem as a follower). The follower’s problem is the same as the customer’s problem (5.4) in the discussions of competitive equilibrium. The main difference here is that the price \( p(t) \) given to the customer is the price selected by the station. This means that the station as a leader has already chosen the price before the customer starts to do the business.

Next, we consider the station’s problem in the Stackelberg competition as a leader.

**Definition 16** (Station’s problem as a leader).

\[
\max_{\dot{x}_S(t), p(t)} \int_0^T (p(t) - r(t)) \dot{x}_S(t) dt, \quad (5.111)
\]

s.t. \( \dot{x}_S(t) \geq 0, x_S(T) = X, x_S(0) = x_0 \), \( (5.112) \)

\[
\dot{x}_S(t) = \arg \max \int_0^T (\alpha - p(t)) \dot{x}(t) - \beta \dot{x}^2(t) dt, \quad (5.113)
\]

s.t. \( (x(t), \dot{x}(t)) \in \mathcal{X}_D \).

\[
\int_0^T (\alpha - p(t)) \dot{x}_S(t) - \beta \dot{x}_S^2(t) dt \geq C. \quad (5.114)
\]
where the constraint (5.113) indicates that the charging profile must also be an optimal solution to the customer’s problem as defined in (5.4). The set $\mathcal{X}_D$ is defined in (5.4).

**Remark 5.5.1.** The constraint (5.114) is imposed to ensure a minimum level of welfare for the customer.

The only difference in the station’s problem as a Stackelberg leader comparing to the station’s problem in the competitive equilibrium case is that the station have an extra constraint (5.113). Note that this constraint complies with the solution to the customer’s problem (5.4). This can be viewed as an extra information about the market price $p(t)$. Although the profile $p(t)$ still remains unchanged and exogenous. The station as a leader has access to the structure of $p(t)$ in terms of the customer’s optimal choice. This plays an important role in the optimal solution of the station and leads to a different result comparing to the competitive equilibrium solution.

**Definition 17** (Stackelberg market equilibrium). The Stackelberg equilibrium is the pair of charging rate profile $\dot{x}(g)(t), t \in [0, T]$ and market price profile $p(g)(t) \in C[0, T]$ such that $\dot{x}(g)(t)$ is the optimal solution to the problem (5.111) and the customer’s problem (5.4) under the market price $p(g)(t) \in C[0, T]$.

Next, we characterize the charging profile at the competitive equilibrium defined in (5.89) in the following result, which relies on the following definitions and notations.

- Denote the trajectory $x(g)(t), \dot{x}(g)(t), t \in [0, T]$ as the solution to the social welfare problem (5.166).
- Define the trajectory $u_i(g)(t)$ over $t = [t_i, T]$ as

$$u_i(g)(t) = \frac{1}{4\beta_i} (\alpha_i - \gamma_i^* - r(t)). \quad (5.115)$$

where $\gamma_i^*$ is a constant parameter to be determined.

Similar to Definition 9, we define the following customer set.
Definition 18. Define the customer set $J_g(t) = \{i \in A(t) | x_i^{(g)}(t) - L_ix_i(0) = 0\}$. Define $J^c_g$ as the complement of $J_g(t): J^c_g(t) = \{1, 2, \ldots, N(t)\}/J_g(t)$, for $t \in [0, T]$ where $A(t)$ is defined in (5.19).

Theorem 5.5.2. Suppose the Stackelberg equilibrium (5.176) exists. The charging rate at the equilibrium is determined by a boundary value problem with differential equation defined as

$$\dot{x}_i^{(g)}(t) = \begin{cases} 
0 & \text{if } t \in [0, t_i), \\
u_i^{(g)}(t) & \text{if } \sum_{i=1}^{N(t)} \nu_i^{(g)}(t) > 0, x_i^{(g)}(t) - L_ix_i,0 > 0, t \in [t_i, T] \\
\frac{1}{4\beta_i} (\alpha_i - \gamma_i^{*}) + \frac{1}{4\beta_k} (\gamma_k^{*} - \alpha_k) \sum_{k \in J^c_g} (t) \in [t_i, T] \\
0 & \text{if } x_i^{(g)}(t) - L_ix_i(0) = 0, t \in [t_i, T], 
\end{cases}$$

(5.116)

for $t \in [0, T]$ and $i = 1, 2, \ldots, N(t)$, where $N(t)$ is defined in (5.19). The customer set $J(t) = \{i \in A(t) | x_i^{*}(t) - L_ix_i^{*}(0) = 0\}$ represents the set of customers which satisfies the state constraint at the time $t$, where $A(t)$ is the set of customers who arrive by the time $t$, defined in (5.19).

The parameter $\gamma_i^{*}$ is defined as $\gamma_i^{*} \triangleq \mu_i^{*} + \lambda_i^{*}$, where $\mu_i^{*}, i = 1, 2, \ldots, N$ are constants that serve as Lagrange multipliers in the solution of the station’s problem (5.166), and the constant $\lambda_i^{*}, i = 1, 2, \ldots, N$ are Lagrange multipliers in the solution of the customers’ problem (5.85). The boundary conditions for the equation (5.178) are $x_i^{(g)}(0) = x_i,0, x_i^{(g)}(T) = X_i$ for $i = 1, \ldots, N(t)$

The parameter $\gamma_i$ is determined by solving the boundary value differential equations (5.178) as the unknown parameters.
At the equilibrium, the market price \( p_i^{(g)}(t) \) is characterized as

\[
p_i^{(g)}(t) = \begin{cases} 
\frac{1}{2}(\alpha_i - \lambda^*_i + \mu^*_i + r(t)) & \text{if } \sum_{i=1}^{N(t)} u_i^{(g)}(t) > 0, x_i^{(g)}(t) - L_i x_{i,0} > 0, \\
\frac{1}{2} \left( \alpha_i - \lambda^*_i + \mu^*_i \right) - \frac{1/2}{\sum_{k \in J}(4\beta_k)} \sum_{k \in J} \frac{1}{4\beta_k} \left( \mu^*_k + \lambda^*_k - \alpha_k \right) & \text{if } \sum_{i=1}^{N(t)} u_i^{(g)}(t) \leq 0, x_i^{(g)}(t) - L_i x_{i,0} > 0, \\
\alpha_i - \lambda^*_i & \text{if } x_i^{(g)}(t) - L_i x_{i,0} = 0, i \in J.
\end{cases}
\]  

(5.117)

for \( \lambda^*_i + \mu^*_i = \gamma^*_i \), where the constant \( \lambda^*_i \) is given by

\[
\lambda^*_i = \frac{1}{X - x_0} \left( C_i - \int_0^T (\dot{x}_i^{(g)}(t))^2 \, dt \right). 
\]  

(5.118)

and \( \mu^*_i = \gamma^*_i - \lambda^*_i \).

Proof. We prove the theorem by considering the solution to the station’s problem (5.166). Using the constraint (5.174), the problem can be rewritten as

\[
\max_{\lambda_i(t), x_i,S(t)} \int_0^T \sum_{i=1}^{N(t)} (\alpha_i - \lambda_i(t) - r(t)) \dot{x}_{i,S}(t) - 2\beta_i \sum_{i=1}^{N(t)} \dot{x}_{i,S}(t)^2 \, dt, 
\]  

(5.119)

s.t.

\[
\sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, 
\]  

(5.120)

\[
x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \text{for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}. 
\]  

(5.121)

\[
x_{i,S}(t) - L_i x_{i,0} \geq 0. 
\]  

(5.122)

To simplify the solving process, we avoid solving the problem over two decision variables. Instead, we first treat the Lagrange multiplier \( \lambda_i(t) \) as given, then we characterize the charging rate profile \( \dot{x}(t) \) and show that the charging rate profile is independent of the Lagrange multiplier \( \lambda_i(t) \). With the charging rate profile obtained at the Stackelberg equilibrium, we characterize \( \lambda_i(t) \) using the constraint on the customer’s
The corresponding optimal control problem is formulated as

\[
\min_{u(t)} \int_0^T \sum_{i=1}^{N(t)} (r(t) + \lambda(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) dt, \tag{5.123}
\]

s.t. \( \forall t \in [0, T], x_{i,S}(T) = X_i, \) \( \sum_{i=1}^{N(t)} u_i(t) \geq 0, \)

\[
\dot{x}_{i,S}(t) = u_i(t), i = 1, 2, \ldots, N(t), \tag{5.126}
\]

\[
L_i x_{i,0} - x_{i,S}(t) \leq 0. \tag{5.127}
\]

We can write the augmented Hamiltonian associated with this optimal control problem as follows.

\[
H = \sum_{i=1}^{N(t)} (r(t) + \lambda^*_i(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) + \sum_{i=1}^{N(t)} \mu_i(t) u_i(t) + \eta_i(t)(Lx_{i,0} - x_{i,S}(t))
\]

where \( \mu_i(t), i = 1, \ldots, N(t) \) is the Lagrange multiplier related to the dynamic equations. The multiplier \( \eta_i(t), i = 1, \ldots, N(t) \) is associated with the state constraint (5.189).

The state equation and costate equation are

\[
\dot{x}_{i,S}^*(t) = \frac{\partial H}{\partial \mu_i} = u_i^*(t),
\]

\[
\dot{\mu}_i^*(t) = -\frac{\partial H}{\partial x_{i,S}} = \eta_i(t).
\]

Thus, \( \mu_i^*(t), i = 1, 2, \ldots, N \) are constants when the state constraint (5.189) is not
binding. The input $u_i(t)$ should minimize the Hamiltonian for any $t \in [t_i, T]$.

$$\min_{u_i(t)} \mathcal{H} = \sum_{i=1}^{N(t)} (r(t) + \lambda_i^*(t) - \alpha_i) u_i(t)$$
$$+ 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) + \sum_{i=1}^{N(t)} \mu_i^*(t) u_i(t) + \eta_i^*(t)(Lx_{i,0} - x_{i,S}(t))$$
$$\text{s.t. } \sum_{i=1}^{N(t)} u_i(t) \geq 0. \quad (5.128)$$

This is a classic constrained quadratic programming problem. We can write the
Lagrangian of this optimization problem as follows.

$$\mathcal{L} = \sum_{i=1}^{N(t)} (r(t) + \lambda_i^*(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t)$$
$$+ \sum_{i=1}^{N(t)} \mu_i(t) u_i(t) - \xi(t) \sum_{i=1}^{N(t)} u_i(t) + \eta_i(t)(Lx_{i,0} - x_{i,S}(t)).$$

where $\xi(t)$ is the Lagrange multiplier associated with the constraint (5.190). At the
optimal solution, the Lagrangian satisfies $\frac{\partial \mathcal{L}}{\partial u_i(t)} = 0$, which implies

$$r(t) - \alpha_i + \lambda_i^*(t) + 4\beta_i u_i^*(t) + \mu_i^*(t) - \xi^*(t) = 0. \quad (5.129)$$

We study the optimal solution based on whether the constraint (5.190) and the
state constraint (5.189) are binding or not.

**Case I:** neither of the constraints (5.189) and (5.190) is binding for any customer
$i = 1, 2, \ldots, N(t)$.

In this case, $\xi^*(t) = 0$, the optimal input $u_i^*(t)$ i.e. the charging rate at the
Stackelberg equilibrium $\dot{x}^{(g)}(t)$ can be obtained by (5.191), which is $\dot{x}^{(g)}(t) = u_i^{(g)}(t)$.

Since neither of the constraint is binding, the multiplier $\lambda_i(t)$ and $\mu_i^*(t)$ are
constants, which implies the parameter $\gamma(t) = \lambda_i^*(t) + \mu_i^*(t)$ is also constant.
By definition, the Stackelberg equilibrium charging rate \( \dot{x}^{(g)}(t) = \dot{x}_{i,D}^{*}(t) \). Using (5.43), the price profile \( p_i^{(g)}(t) \) at Stackelberg equilibrium is given as

\[
p_i(t) = \frac{1}{2}(\alpha_i - \lambda_i^* + \mu_i^* + r(t)). \tag{5.130}
\]

**Case II:** The state constraint (5.189) is not binding for any customer, the sum constraint (5.190) is binding for all customers.

Since constraint (5.190) is binding, we have \( \sum_{k=1}^{N(t)} u_k(t) = 0 \). The multiplier \( \xi^*(t) \) can be obtained by summing up (5.191) over all customers.

\[
\xi^*(t) = \frac{1}{\sum_{k=1}^{N(t)} (4\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{4\beta_k} (r(t) + \mu_k^* - \alpha_k + \lambda_k^*). \tag{5.131}
\]

Substituting (5.193) into (5.191), the charging rate at Stackelberg equilibrium is given by

\[
\dot{x}^{(g)}(t) = \frac{1}{4\beta_i} (\alpha_i - \lambda_i^* - \mu_i^*) + \frac{(4\beta_i)^{-1}}{\sum_{k=1}^{N(t)} (4\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{4\beta_k} (\mu_k^* - \alpha_k + \lambda_k^*). \tag{5.132}
\]

In this case, \( \lambda_i^*(t) \) and \( \mu_i^*(t) \) are both constants since the state constraint (5.189) is not binding. Thus, the parameter \( \gamma_i(t), i = 1, \ldots, N(t) \) is also constant. We defer the discussion of the price profile at the Stackelberg equilibrium in this case to Case III.

**Case III:** The state constraint (5.189) is binding for some customers given that the sum constraint (5.190) for all customers is also binding.

Denote the set of customers who have state constraint (5.189) binding as \( J \) and other customers belong to the complement \( J^c \). For the customers in \( J \), the input must be zero to meet the state constraint, which is imposed by the solution to customer’s
problem. We have

\[ \dot{x}_j^{(g)}(t) = u_i^*(t) = 0, \quad j \in J_g. \] (5.133)

since the constraint (5.190) is also binding, for the customers \( i \in J_g^c \), the inputs must satisfy \( \sum_{i \in J_g^c} u_i^*(t) = 0 \). The optimal inputs can be obtained by summing (5.191) over \( i \in J_g^c \).

\[ \dot{x}_j^{(g)}(t) = u_i^*(t) = \frac{1}{4\beta_i} (\alpha_i - \lambda_i^*(t) - \mu_i^*) \sum_{k \in J_g^c} \frac{1}{4\beta_k} (\mu_k^* - \alpha_k + \lambda_k^*(t)). \] (5.134)

for \( i \in J_g^c \). In this case, the multiplier \( \xi^*(t) \) can be obtained as

\[ \xi^*(t) = \sum_{i \in J_g^c} \frac{1}{4\beta_i} (\alpha_i - \lambda_i^*(t) - \mu_i^*) \sum_{k \in J_g^c} \frac{1}{4\beta_k} (\mu_k^* + \lambda_k^* - \alpha_k). \] (5.135)

Since the state constraint (5.189) for customers \( k \in J_g^c \) is not binding, both \( \mu_k^* \) and \( \lambda_k^* \) are constants. It also implies that \( \gamma_k^* = \mu_k^* + \lambda_k^* \) is also constant. Next we consider the price profile for customers in the set \( J_g^c \). Using (5.196), (5.43) and \( \dot{x}_j^{(g)}(t) = \dot{x}_i^{*,D}(t) \), we obtain the price profile as follows.

\[ p_j^{(g)}(t) = \frac{1}{2} (\alpha_j - \lambda_j^*(t) + \mu_j^*) \sum_{k \in J_g^c} \frac{1}{4\beta_k} (\mu_k^* + \lambda_k^* - \alpha_k). \] (5.136)

for \( j \in J_g^c \).

Note that if \( J^c = \emptyset \), then the price profile is the Stackelberg equilibrium price for the case II.

For the customers \( j \in J_g \), the Lagrange multipliers \( \mu_j^*(t), \quad j \in J_g \) can be obtained
by (5.191) with $u_j^*(t) = 0$ and also (5.197). This is given as

$$
\mu_j^*(t) = \xi^*(t) - (r(t) - \alpha_j + \lambda_j^*(t))
= \frac{1}{\sum_{k \in J_g}(4\beta_k)^{-1}} \sum_{k \in J_g} \frac{1}{4\beta_k} (\mu_k^* + \lambda_k^* - \alpha_k) - (\lambda_j^*(t) - \alpha_j)
$$

(5.137)

for any $j \in J_g$. We can obtain the parameter $\gamma_j(t) = \mu_j^*(t) + \lambda_j^*(t)$,

$$
\gamma_j(t) = \frac{1}{\sum_{k \in J_g}(4\beta_k)^{-1}} \sum_{k \in J_g} \frac{1}{4\beta_k} (\gamma_k - \alpha_k) + \alpha_j.
$$

(5.138)

In this case, the parameter $\gamma_j, j \in J_g$ depends on the parameters of the customer set $J_g^c$. Note that for customers $k \in J_g^c$, $\gamma_k$ is constant since their state constraint is not binding. Thus, $\gamma_j, j \in J_g$ is also constant. The multipliers $\mu^*(t) \geq 0, \lambda_j^*(t) \geq 0$ since the state constraint is binding for $j \in J_g$ in this case. The multiplier $\mu_j^*(t)$ and $\lambda_j^*(t)$ can take any forms that satisfy $\gamma_j^* = \mu_j^*(t) + \lambda_j^*(t)$. One candidate for $\mu_j^*(t)$ and $\lambda_j^*(t)$ are constants values such that $\gamma_j^* = \mu_j^* + \lambda_j^*$.

For customers $j \in J_g$, to achieve Stackelberg equilibrium, the price profile must take appropriate forms such that $\dot{x}_{i,D}^*(t) = 0$. Using (5.43), the profile of price is given as

$$
p_i^{(g)}(t) = \alpha_i - \lambda_i^*.
$$

(5.139)

**Case IV:** the sum constraint (5.190) is not binding while the state constraint (5.189) is binding for every customer. This case never happens because when (5.189) holds for every customer, every charging rate at Stackelberg equilibrium remains zero, which implies that the sum constraint (5.190) is binding.

Summarizing all the four cases discussed above, we can characterize the charging rate and the price profile at the Stackelberg equilibrium in (5.178).

Next, we move on to solve the price profile $p_i^{(g)}(t)$. With the charging rate profile
\( \dot{x}_i^{(g)}(t) \) determined, it remains to determine \( p_i(t) \) and \( \lambda_i(t) \). We recast the station’s problem as follows.

\[
\begin{align*}
\max_{p_i(t), i=\{1,2,\ldots,N(t)\}} \int_0^T \sum_{i=1}^{N(t)} (p_i(t) - r(t)) \dot{x}_i^{(g)}(t) \, dt.
\end{align*}
\]

s.t. \[
\int_0^T (\alpha_i - p_i(t)) \dot{x}_i^{(g)}(t) - \beta \dot{x}_i^{(g)}(t)^2(t) \, dt \geq C_i, \quad i = 1, 2, \ldots, N(t). \quad (5.140)
\]

Since \( \dot{x}_i^{(g)}(t), r(t) \) are given, we can model the scalar \( \int_0^T p_i(t) \dot{x}_i^{(g)}(t) \, dt \) as a decision variable, denoted as \( s_i \). The optimization problem left with us is a linear program with \( N \) variables.

\[
\begin{align*}
\max_{p_i(t), i=\{1,2,\ldots,N(t)\}} \sum_{i=1}^{N(t)} s_i - \int_0^T r(t) \dot{x}_i^{(g)}(t) \, dt.
\end{align*}
\]

s.t. \[
s_i \leq \int_0^T (\alpha_i \dot{x}_i^{(g)}(t) - \beta \dot{x}_i^{(g)}(t)^2(t) - C_i, \quad i = 1, 2, \ldots, N(t). \quad (5.141)
\]

At the optimal solution, the equality of each constraint holds. This implies that at Stackelberg equilibrium, the utility of the customer achieves the lower bound.

Consider the utility of any customer \( i \), at Stackelberg equilibrium, of the customer \( i \). Using (5.43) and note that the utility reaches the lower bound \( C_i \), the utility can be written as

\[
\begin{align*}
\int_0^T \lambda^*_i \dot{x}_i^{(g)}(t) + \beta_i (\dot{x}_i^{(g)}(t))^2 dt = \lambda^*_i (X - x_0) + \int_0^T (\dot{x}_i^{(g)}(t))^2 dt = C_i. \quad (5.142)
\end{align*}
\]

Thus, we can characterize \( \lambda^*_i \) as (5.180).

Next, we will evaluate the social welfare of the Stackelberg equilibrium and make comparison with the social welfare achieved by the competitive equilibrium, which is social optimal. Before all the discussions, we need the following technical lemma.
Lemma 5.5.3. Given the customer’s model (5.4) and station’s model (5.2), the charging rate profile at the competitive equilibrium $\dot{x}^*(t)$ and the charging rate profile at the Stackelberg equilibrium $\dot{x}^{(g)}(t)$. Define the time domain $S$ such that for any time $t \in E$, the charging profile of the competitive equilibrium $\dot{x}^*(t) > 0$. Similarly, define the time domain $S$ such that for any time $t \in E$, the charging profile at the Stackelberg equilibrium $\dot{x}^{(g)}(t) > 0$. The following inequality holds.

$$E \subseteq S.$$  \hfill (5.143)

Moreover, the following inequality holds.

$$0 \leq \gamma^*_0 - \gamma^*_1 \leq \frac{2\beta}{l_e} (X - x_0).$$  \hfill (5.144)

where $0 < l_e \leq T$ is the length of time such that $\dot{x}^*(t) > 0$ over the time period $[0,T]$, i.e. $l_e = E$.

Proof. The charging rate at the Stackelberg equilibrium can be written as

$$\dot{x}^{(g)}(t) = \frac{1}{4\beta} (\alpha - \gamma^*_1 - r(t))^+.$$  \hfill (5.145)

where $\gamma^*_1 = \mu^*_1 + \lambda^*_1$.

On the other hand, the charging rate at the competitive equilibrium can be written as

$$\dot{x}^*(t) = \frac{1}{2\beta} (\alpha - \gamma^*_0 - r(t))^+.$$  \hfill (5.146)

where $\gamma^*_0 = \mu^*_0 + \lambda^*_0$. Define the time domain $S$ such that for any time $t \in E$, the charging profile of the competitive equilibrium $\dot{x}^*(t) > 0$. Similarly, define the time domain $S$ such that for any time $t \in E$, the charging profile at the Stackelberg
equilibrium \( \dot{x}^{(g)}(t) > 0 \).

\[
\int_0^T \dot{x^*}(t) dt = \int_E \frac{1}{2\beta} (\alpha - \gamma_0^* - r(t)) dt = X - x_0. \tag{5.147}
\]

\[
\int_0^T \dot{x}^{(g)}(t) dt = \int_S \frac{1}{4\beta} (\alpha - \gamma_1^* - r(t)) dt = X - x_0. \tag{5.148}
\]

Suppose there exists a time \( t \in [0, T] \), such that \( \dot{x}^{(g)}(t) < 0 \) and \( \dot{x^*}(t) > 0 \). This implies that \( \gamma_1^* > \gamma_0^* \). Suppose there exists another time \( t' \neq t \) such that \( \dot{x}^{(g)}(t') > 0 \) and \( \dot{x^*}(t) < 0 \). But this will leads to \( \gamma_1^* < \gamma_0^* \), which is contradictory. Thus, we can see that once we suppose that \( \dot{x}^{(g)}(t) < 0 \) and \( \dot{x^*}(t) > 0 \) occurs at some \( t \), it implies that \( S \subset E \). In this case, at time \( t \in S \cap E = S \), we can see that \( \dot{x}^{(g)}(t) < \dot{x^*}(t) \) because \( \gamma_1^* > \gamma_0^* \) and both charging rates are positive. Thus, we have

\[
\int_{E \cap S} \dot{x^*}(t) dt > \int_S \dot{x}^{(g)}(t) dt.
\]

Moreover, the integral of the charging rate profile at the competitive equilibrium can be written as

\[
\int_{E} \dot{x^{*}}(t) dt = \int_S \dot{x^{*}}(t) dt + \int_{E \setminus S} \dot{x^{*}}(t) dt.
\]

Since \( \int_{E \setminus S} \dot{x^{*}}(t) dt > 0 \), it implies that

\[
\int_{E} \dot{x^{*}}(t) dt > \int_S \dot{x}^{(g)}(t) dt. \tag{5.149}
\]

However, (5.149) contradicts (5.147). Thus, we know that there does not exist any time \( t \in [0, T] \), such that \( \dot{x}^{(g)}(t) < 0 \) and \( \dot{x^*}(t) > 0 \). This implies \( E \subseteq S \) and also \( \gamma_0 > \gamma_1 \).

With the relation between the sets \( E \) and \( S \), the integral of the Stackelberg charg-
ing rate can be rewritten as

\[
\int_S \dot{x}(g)(t)dt = \int_{\mathcal{E}} \dot{x}(g)(t)dt + \int_{\mathcal{S}/\mathcal{E}} \dot{x}(g)(t)dt
\]

\[
= \int_{\mathcal{E}} \frac{1}{4\beta} (\alpha - \gamma^*_0 - r(t) + \gamma^*_0 - \gamma^*_1) dt + \int_{\mathcal{S}/\mathcal{E}} \dot{x}(g)(t)dt
\]

\[
= \frac{1}{2} \int_{\mathcal{E}} \dot{x}^*(t)dt + |\mathcal{E}|(\gamma^*_0 - \gamma^*_1)/(4\beta) + \int_{\mathcal{S}/\mathcal{E}} \dot{x}(g)(t)dt
\]

\[
\geq \frac{1}{2}(X - x_0) + |\mathcal{E}|(\gamma^*_0 - \gamma^*_1)/(4\beta).
\]  

(5.150)

The equality holds when \(\mathcal{E} = \mathcal{S}\). Further, we can simplify (5.150) and obtain

\[
\gamma^*_0 - \gamma^*_1 \leq \frac{2\beta}{|\mathcal{E}|}(X - x_0).
\]  

(5.151)

Combining \(\gamma_0 > \gamma_1\), we obtain (5.144).

\[\square\]

**Lemma 5.5.4.** Given the model of the station and customer, if the Lagrange multiplier \(\mu_1 = \mu_0\), the utility achieved by the station at the Stackelberg equilibrium (denoted as \(K_s^{(g)}\)) is greater than the utility achieved by the station at the competitive equilibrium (denoted as \(K^*_s\)). The lower bound on the difference between the two utilities is given as

\[
K_s^{(g)} - K^*_s > \frac{1}{8\beta}(\gamma_0 - \gamma_1)^2 l_e.
\]  

(5.152)

where \(0 < l_e < T\) is the length of the time interval \(\mathcal{E} = \{t|\dot{x}^*(t) > 0\}\).

**Proof.** First consider \(K_s^{(g)}\), the station’s utility achieved at the Stackelberg equilibrium. Using the price profile (??) obtained at the Stackelberg equilibrium, the
station’s optimal utility can be represented as

\[ K_s^{(g)} = \int_0^T (p(t) - r(t)) \dot{x}^{(g)}(t) - \beta (\dot{x}^{(g)}(t))^2 dt \]

\[ = 2\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt + \mu_1^*(X - x_0). \tag{5.153} \]

The station’s optimal utility achieved at competitive equilibrium is

\[ K^*_s = \mu_0^*(X - x_0). \]

By Lemma 5.5.3, the integral \( \int_0^T (\dot{x}^{(g)}(t))^2 dt \) satisfies

\[ \int_0^T (\dot{x}^{(g)}(t))^2 dt \geq \frac{1}{4} \int_\mathcal{E} (\dot{x}^*(t))^2 dt \]

\[ + \frac{1}{2\beta} (\gamma_0 - \gamma_1)(X - x_0) + \frac{1}{16\beta^2} (\gamma_0 - \gamma_1)^2 |\mathcal{E}|. \tag{5.154} \]

where the set \( \mathcal{E} = \{ t \in [0, T] | \dot{x}^*(t) > 0 \} \).

Since we suppose \( \mu_0^* = \mu_1^* \), the gap between \( K_s^{(g)} \) and \( K^*_s \) can be lower bounded as

\[ K_s^{(g)} - K^*_s = 2\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt \geq \frac{\beta}{2} \int_\mathcal{E} (\dot{x}^*(t))^2 dt + \frac{1}{8\beta} (\gamma_0 - \gamma_1)^2 |\mathcal{E}| \]

\[ > \frac{1}{8\beta} (\gamma_0 - \gamma_1)^2 |\mathcal{E}|. \]

\[ \square \]

Remark 5.5.5. Note that the \( p^*(t) = \max\{r(t) + \mu_0^*, \alpha - \lambda_0^* \} \) at the competitive equilibrium and the price profile at the Stackelberg equilibrium is the average of the two terms \( r(t) + \mu_1^*, \alpha_1^* - \lambda_1^* \). We compare the station’s utilities obtained in the two cases based on the criterion that \( \mu_0^* = \mu_1^* \). By Lemma 5.5.4 it can be seen that the station is better off in the Stackelberg case.
Example 11. Consider the charging business between one station and a single customer in the Stackelberg case. Suppose the wholesale electricity price \( r(t) = \sin(t) + 1.1 \). The customer’s preference \( \alpha = 2, \beta = .1 \). The charging rate and SoC at the Stackelberg equilibrium are illustrated in Figure 5.8.

In this example, the Lagrange multiplier \( \gamma_1 = \lambda_1^* + \mu_1^* = 0.4391 \). Assume that the lower bound of the customer’s utility is \( C = 15.7078 \), then we obtain the multipliers \( \lambda_1^* = 0.3000, \mu_1^* = 0.1391 \), we plot the market price in this case in Figure 5.9.

Since the competitive equilibrium is socially optimal, it is interesting to evaluate
Figure 5.9. Charging rate and market price at the Stackelberg equilibrium
the the gap between social welfare of the Stackelberg equilibrium and the optimal social welfare under various situations. Notice that the social optimality depends on the electric price signal \( r(t) \) given parameters \( \alpha, \beta \) fixed. Thus, we conjecture that the social welfare gap of the market equilibrium also depends on the price signal \( r(t) \).

**Proposition 5.5.6.** Given the station and customer model in (5.2) and (5.4) and also given a price \( p^*(t), t \in [0, T] \) at the competitive equilibrium. Denote the customer’s utility at the competitive equilibrium under the price \( p^*(t) \) as \( C \). Consider the Stackelberg problem with the constraint that the customer earns the same utility \( C \). Define the gap between the social welfare at the competitive equilibrium \( W_c \) and the social welfare at the Stackelberg equilibrium \( W_g \) as \( \Delta W = W_c - W_g \). This social optimality gap satisfies the following inequality.

\[
\Delta W \leq \left(\frac{2\beta}{l_e} - \frac{\beta}{T}\right) (X - x_0)^2. \tag{5.155}
\]

where \( 0 < l_e \leq T \) is the length of time such that the charging rate at the competitive equilibrium \( \dot{x}^*(t) > 0 \) over the time period \([0, T]\).

**Proof.** Given that the customer’s utility is equal to \( C > 0 \), the social welfare of the competitive equilibrium and Stackelberg equilibrium can be represented as

\[
W_c = K_s^* + C, \quad W_g = K_s^{(g)} + C. \tag{5.156}
\]

The social optimality gap can be represented as

\[
\Delta S = K_s^* - K_s^{(g)} = (\mu_0^* - \mu_1^*)(X - x_0) - 2\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt. \tag{5.157}
\]
By (5.144) and \( \gamma_0 = \mu_0^* + \lambda_0^*, \gamma_1 = \mu_1^* + \lambda_1^* \), we know that

\[
\mu_0^* - \mu_1^* \leq \frac{2\beta}{l_e}(X - x_0) - (\lambda_0^* - \lambda_1^*). \tag{5.158}
\]

Denote the customer’s utility at the competitive equilibrium as \( K_c^* \), which can be written as

\[
K_c^* = \beta \int_0^T (\dot{x}^*(t))^2 dt + \lambda_0(X - x_0). \tag{5.159}
\]

At the Stackelberg equilibrium, the customer’s utility is

\[
K_c^{(g)} = 3\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt + \lambda_1(X - x_0). \tag{5.160}
\]

Combining the representation of \( K_c^* \) and \( K_c^{(g)} \), we obtain

\[
(\lambda_0^* - \lambda_1^*)(X - x_0) = 3\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt - \beta \int_0^T (\dot{x}^*(t))^2 dt. \tag{5.161}
\]

Thus,

\[
(\mu_0^* - \mu_1^*)(X - x_0) = \frac{2\beta}{l_e}(X - x_0)^2 - 3\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt - \beta \int_0^T (\dot{x}^*(t))^2 dt. \tag{5.162}
\]
By (5.158), the social optimality gap satisfies
\[
\Delta S \leq \frac{2\beta}{l_e} (X - x_0)^2 - 5\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt + \beta \int_0^T (\dot{x}^*(t))^2 dt
\]
\[
= \frac{2\beta}{l_e} (X - x_0)^2 - \beta \int_0^T (\dot{x}^{(g)}(t))^2 dt - 4\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt
\]
\[
+ \beta \int_0^T (\dot{x}^*(t))^2 dt.
\]

By Cauchy-Schwartz inequality, we obtain
\[
\Delta S \leq \frac{2\beta}{l_e} (X - x_0)^2 - \beta (X - x_0)^2
\]
\[
- 4\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt + \beta \int_0^T (\dot{x}^*(t))^2 dt.
\]
(5.163)

By (5.154), we obtain
\[
- 4\beta \int_0^T (\dot{x}^{(g)}(t))^2 dt + \beta \int_0^T (\dot{x}^*(t))^2
\]
\[
\leq -2(\gamma_0 - \gamma_1)(X - x_0) - \frac{1}{4\beta} (\gamma_0 - \gamma_1)^2 l_e dt
\]
(5.164)

Substitute the inequality into the social optimality gap, we obtain
\[
\Delta S \leq \left( \frac{2\beta}{l_e} - \frac{\beta}{T} \right) (X - x_0)^2 - 2(X - x_0)(\gamma_0 - \gamma_1)
\]
\[
- \frac{1}{4\beta} (\gamma_0 - \gamma_1)^2 l_e.
\]
(5.165)

Recall that \( \gamma_0 - \gamma_1 > 0 \) from Lemma 5.5.3, we obtain the upper bound of the social optimality gap (5.155).

**Remark 5.5.7.** By (5.155), we can see that the gap would become zero if the parameter \( \beta = 0 \) which is related to the quadratic cost term of the charging rate in the customer’s utility (5.4). This implies that the source of the inefficiency comes from the cycling
5.5.1 The Stackelberg Equilibrium of the Case with Multiple Customers in a Batch Arrival

In this section, we consider Stackelberg equilibrium in the batch arrival case. We consider the scenario that multiple customers arrive at the station as a batch with their arrival times known to the station. In other words, during a given time interval \([0, T]\), there are \(N\) PEV customers that arrive at the station at given times \(0 \leq t_1 \leq t_2 \leq \ldots t_N < T\). The station knows the exact arrival times \(t_1, t_2, \ldots, t_N\) and then it can generate the optimal charging rate based on the whole batch of PEV customers. In this model, the Assumption 12 holds. The customer’s model is the same as the model defined in Definition 11.

**Definition 19** (The station’s Stackelberg problem for multiple customers of batch arrival). The station’s Stackelberg problem in the batch arrival case is defined as

\[
\begin{align*}
&\max_{\dot{x}_{i,S}(t), p_i(t), i=\{1,2,\ldots,N(t)\}} \int_0^T \sum_{i=1}^{N(t)} \left( p_i(t) - r(t) \right) \dot{x}_{i,S}(t) \, dt. \\
\text{s.t.} \quad \sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, \\
x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(0) = 0, &\text{for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}, \\
\dot{x}_{i,S}(t) = \arg \min_{\dot{x}_i(t)} \int_0^T (p_i(t) - \alpha_i) \dot{x}_i(t) + \beta_i \dot{x}_i^2(t) \, dt. \\
\text{s.t.} \quad x_i(t) - L_i x_i(0) \geq 0, \\
x_i(0) = x_{i,0}, \dot{x}_i(t) = 0, &\text{for } t \in [0, t_i], x_i(T) = X_{i,S}. \\
\int_0^T (p_i(t) - \alpha_i) \dot{x}_{i,S}(t) + \beta_i \dot{x}_{i,S}^2(t) \, dt \geq C_i.
\end{align*}
\]

where \(N(t) = |\{i = 1, 2, \ldots, N|t_i \leq t\}|\) is the number of the customers who have arrived at the station by the time \(t\). The constraint \((5.167)\) comes from the Assumption 12.
The constraint (5.169) means that the charging rate \( \dot{x}_{i,S}(t) \) is also an optimal solution to the customer’s problem. For convenience, the constraint (5.169) can be further substituted by the solution to the customer’s problem. Then, the station’s problem can be written as

\[
\max_{\dot{x}_{i,S}(t), p_i(t), \lambda^*_i(t)} \int_0^T \sum_{i=1}^{N(t)} (p_i(t) - r(t)) \dot{x}_{i,S}(t) \, dt.
\]

(5.171)

s.t. \( \sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0, \)

(5.172)

\( x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \) for \( t \in [0, t_i], x_{i,S}(T) = X_{i,S}, \)

(5.173)

\( \dot{x}_{i,S}(t) = \frac{1}{2\beta_i} (\alpha_i - \lambda^*_i(t) - p_i(t)), \) \( t \in [t_i, T] \)

(5.174)

\( x_{i,S}(t) - L_i x_{i,0} \geq 0. \)

(5.175)

Note that the Lagrange multiplier \( \lambda^*_i(t) \) is derived in the customer’s problem such that the constraint \( x_{i,S}(t) - L_i x_{i,0} \geq 0 \) is satisfied. The multiplier \( \lambda^*_i(t) \) is also a decision variable of the optimization problem because given any price profile \( p_i(t) \), the solution of the customer’s problem generates a multiplier \( \lambda^*_i(t) \).

**Definition 20** (Stackelberg equilibrium with multiple customers in a batch). The Stackelberg equilibrium for the station’s problem (5.166) in a batch arrival case is defined as a set of charging profiles \( \{\dot{x}^{(g)}_{i}(t)\}_{i=1}^{N} \) defined over \([0, T]\) such that

\[
\dot{x}^{(g)}_{i}(t) = \dot{x}^*_{i,D}(t) = \dot{x}^*_{i,S}(t), \quad t \in [0, T],
\]

(5.176)

for each \( i = 1, \ldots, N \). where \( \dot{x}^*_{i,D}(t) \) is the optimal solution to the customer’s problem (5.85) and \( \dot{x}^*_{i,S}(t) \) is the optimal solution to (5.85 the station’s problem (5.166).

Next, we characterize the charging profile at the competitive equilibrium defined in (5.89) in the following result, which relies on the following definitions and notations.
• Denote the trajectory $x^{(g)}(t), \dot{x}^{(g)}(t), t \in [0,T]$ as the solution to the social welfare problem (5.166).

• Define the trajectory $u^{(g)}_i(t)$ over $t = [t_i, T]$ as

$$u^{(g)}_i(t) = \frac{1}{4j_i} \left( \alpha_i - \gamma_i^* - r(t) \right). \quad (5.177)$$

where $\gamma_i^*$ is a constant parameter to be determined.

Similar to Definition 9, we define the following customer set.

**Definition 21.** Define the customer set $J^c_g(t) = \{ i \in A(t) | x^{(g)}_i(t) - L_i x_i(0) = 0 \}$.

Define $J^c_g$ as the complement of $J_g(t): J^c_g(t) = \{1,2,\ldots,N(t)\}/J_g(t)$, for $t \in [0,T]$ where $A(t)$ is defined in (5.19).

**Theorem 5.5.8.** Suppose the Stackelberg equilibrium (5.176) exists. The charging rate at the equilibrium is determined by a boundary value problem with differential equation defined as

$$\dot{x}^{(g)}_i(t) = \begin{cases} 
0 & \text{if } t \in [0,t_i), \\
u^{(g)}_i(t) & \text{if } \sum_{i=1}^{N(t)} u^{(g)}_i(t) > 0, x^{(g)}_i(t) - L_i x_i(0) > 0, t \in [t_i, T] \\
\frac{1}{4j_i} \left( \alpha_i - \gamma_i^* \right) + \frac{(4j_i \gamma_i^*)^{-1}}{\sum_{k \in J^c_g} \sum_{k \in J_g} \frac{1}{4j_k} \left( \gamma_k^* - \alpha_k \right)} t \in [t_i, T] \\
& \text{if } \sum_{i=1}^{N(t)} u^{(g)}_i(t) = 0, x^{(g)}_i(t) - L_i x_i(0) > 0, t \in [t_i, T] \\
0 & \text{if } x^{(g)}_i(t) - L_i x_i(0) = 0, t \in [t_i, T],
\end{cases} \quad (5.178)$$

for $t \in [0,T]$ and $i = 1,2,\ldots,N(t)$, where $N(t)$ is defined in (5.19). The customer set $J(t) = \{ i \in A(t) | x^*_i(t) - L_i x^*_i(0) = 0 \}$ represents the set of customers which satisfies the state constraint at the time $t$, where $A(t)$ is the set of customers who arrive by the time $t$, defined in (5.19).

The parameter $\gamma_i^*$ is defined as $\gamma_i^* \triangleq \mu_i^* + \lambda_i^*$, where $\mu_i^*, i = 1,2,\ldots,N$ are constants that serve as Lagrange multipliers in the solution of the station’s problem (5.166), and the constant $\lambda_i^*, i = 1,2,\ldots,N$ are Lagrange multipliers in the solution of the

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customers’ problem (5.85). The boundary conditions for the equation (5.178) are 
\[x_i^{(g)}(0) = x_{i,0}, x_i^{(g)}(T) = X_i\] for \(i = 1, \ldots, N(t)\)

The parameter \(\gamma_i\) is determined by solving the boundary value differential equations (5.178) as the unknown parameters.

At the equilibrium, the market price \(p_i^{(g)}(t)\) is characterized as

\[
p_i^{(g)}(t) = \begin{cases} 
\frac{1}{2}(\alpha_i - \lambda_i^* + r(t)) & \text{if } \sum_{i=1}^{N(t)} u_i^{(g)}(t) > 0, x_i^{(g)}(t) - L_i x_{i,0} > 0, \\
\frac{1}{2}(\alpha_i - \lambda_i^* + \mu_i^* - \frac{1}{2\beta_i} \sum_{k \in J} \frac{1}{4\beta_k} (\mu_k + \lambda_k^* - \alpha_k) & \text{if } \sum_{i=1}^{N(t)} u_i^{(g)}(t) \leq 0, x_i^{(g)}(t) - L_i x_{i,0} > 0, \\
\alpha_i - \lambda_i^* & \text{if } x_i^{(g)}(t) - L_i x_{i,0} = 0, i \in J.
\end{cases}
\] (5.179)

for \(\lambda_i^* + \mu_i^* = \gamma_i^*\), where the constant \(\lambda_i^*\) is given by

\[
\lambda_i^* = \frac{1}{X - x_0} \left( C_i - \int_0^T (\dot{x}_i^{(g)}(t))^2 dt \right).
\] (5.180)

and \(\mu_i^* = \gamma_i^* - \lambda_i^*\).

Proof. We prove the theorem by considering the solution to the station’s problem (5.166). Using the constraint (5.174), the problem can be rewritten as

\[
\max_{\lambda_i(t), \dot{x}_{i,S}(t)} \int_0^T \sum_{i=1}^{N(t)} (\alpha_i - \lambda_i(t) - r(t)) \dot{x}_{i,S}(t) - 2\beta_i \sum_{i=1}^{N(t)} \dot{x}_{i,S}^2(t) dt, 
\] (5.181)

s.t. \(\sum_{i=1}^{N(t)} \dot{x}_{i,S}(t) \geq 0,\) (5.182)

\[x_{i,S}(0) = x_{i,0}, \dot{x}_{i,S}(t) = 0, \text{for } t \in [0, t_i], x_{i,S}(T) = X_{i,S}.\] (5.183)

\[x_{i,S}(t) - L_i x_{i,0} \geq 0.\] (5.184)

To simplify the solving process, we avoid solving the problem over two decision vari-
ables. Instead, we first treat the Lagrange multiplier $\lambda_i(t)$ as given, then we characterize the charging rate profile $\dot{x}(t)$ and show that the charging rate profile is independent of the Lagrange multiplier $\lambda_i(t)$. With the charging rate profile obtained at the Stackelberg equilibrium, we characterize $\lambda_i(t)$ using the constraint on the customer’s utility.

The corresponding optimal control problem is formulated as

$$
\min_{u(t)} \int_0^T \sum_{i=1}^{N(t)} (r(t) + \lambda_i(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t)dt,
$$

(5.185)

s.t. $\forall t \in [0, T]$, $x_{i,S}(T) = X_i$,

(5.186)

$$
\sum_{i=1}^{N(t)} u_i(t) \geq 0,
$$

(5.187)

$$
\dot{x}_{i,S}(t) = u_i(t), i = 1, 2, \ldots, N(t),
$$

(5.188)

$$
L_i x_{i,0} - x_{i,S}(t) \leq 0.
$$

(5.189)

We can write the augmented Hamiltonian associated with this optimal control problem as follows.

$$
\mathcal{H} = \sum_{i=1}^{N(t)} (r(t) + \lambda_i^*(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) + \sum_{i=1}^{N(t)} \mu_i(t) u_i(t) + \eta_i(t)(L x_{i,0} - x_{i,S}(t))
$$

where $\mu_i(t), i = 1, \ldots, N(t)$ is the Lagrange multiplier related to the dynamic equations. The multiplier $\eta_i(t), i = 1, \ldots, N(t)$ is associated with the state constraint (5.189).

The state equation and costate equation are

$$
\dot{x}_{i,S}^*(t) = \frac{\partial \mathcal{H}}{\partial \mu_i} = u_i^*(t),
$$

$$
\dot{\mu}_i^*(t) = -\frac{\partial \mathcal{H}}{\partial x_{i,S}^*} = \eta_i(t).
$$

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Thus, $\mu^*_i(t), i = 1, 2, \ldots, N$ are constants when the state constraint (5.189) is not binding. The input $u_i(t)$ should minimize the Hamiltonian for any $t \in [t_i, T]$.

$$\min_{u_i(t)} \mathcal{H} = \sum_{i=1}^{N(t)} (r(t) + \lambda^*_i(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) + \sum_{i=1}^{N(t)} \mu^*_i(t) u_i(t)$$

$$+ \eta^*_i(t)(Lx_{i,0} - x_{i,S}(t))$$

$$\text{s.t. } \sum_{i=1}^{N(t)} u_i(t) \geq 0. \quad (5.190)$$

This is a classic constrained quadratic programming problem. We can write the Lagrangian of this optimization problem as follows.

$$\mathcal{L} = \sum_{i=1}^{N(t)} (r(t) + \lambda^*_i(t) - \alpha_i) u_i(t) + 2\beta_i \sum_{i=1}^{N(t)} u_i^2(t) + \sum_{i=1}^{N(t)} \mu^*_i(t) u_i(t)$$

$$- \xi(t) \sum_{i=1}^{N(t)} u_i(t) + \eta_i(t)(Lx_{i,0} - x_{i,S}(t)).$$

where $\xi(t)$ is the Lagrange multiplier associated with the constraint (5.190). At the optimal solution, the Lagrangian satisfies $\frac{\partial \mathcal{L}}{\partial u_i(t)} = 0$, which implies

$$r(t) - \alpha_i + \lambda^*_i(t) + 4\beta_i u^*_i(t) + \mu^*_i(t) - \xi^*(t) = 0. \quad (5.191)$$

We study the optimal solution based on whether the constraint (5.190) and the state constraint (5.189) are binding or not.

**Case I:** neither of the constraints (5.189) and (5.190) is binding for any customer $i = 1, 2, \ldots, N(t)$.

In this case, $\xi^*(t) = 0$, the optimal input $u^*_i(t)$ i.e. the charging rate at the Stackelberg equilibrium $\dot{x}^{(g)}(t)$ can be obtained by (5.191), which is $\dot{x}^{(g)}(t) = u^{(g)}_i(t)$.

Since neither of the constraint is binding, the multiplier $\lambda^*(t)$ and $\mu^*_i(t)$ are constants, which implies the parameter $\gamma(t) = \lambda^*_i(t) + \mu^*_i(t)$ is also constant.
By definition, the Stackelberg equilibrium charging rate \( \dot{x}^{(g)}(t) = \dot{x}_{i,D}^*(t) \). Using (5.43), the price profile \( p_i^{(g)}(t) \) at Stackelberg equilibrium is given as
\[
p_i(t) = \frac{1}{2} (\alpha_i - \lambda_i^* + \mu_i^* + r(t)). \tag{5.192}
\]

**Case II:** The state constraint (5.189) is not binding for any customer, the sum constraint (5.190) is binding for all customers.

Since constraint (5.190) is binding, we have \( \sum_{k=1}^{N(t)} u_k(t) = 0 \). The multiplier \( \xi^*(t) \) can be obtained by summing up (5.191) over all customers.

\[
\xi^*(t) = \frac{1}{\sum_{k=1}^{N(t)} (4\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{4\beta_k} (r(t) + \mu_k^* - \alpha_k + \lambda_k^*). \tag{5.193}
\]
Substituting (5.193) into (5.191), the charging rate at Stackelberg equilibrium is given by
\[
\dot{x}^{(g)}(t) = \frac{1}{4\beta_i} (\alpha_i - \lambda_i^* - \mu_i^*) + \frac{(4\beta_i)^{-1}}{\sum_{k=1}^{N(t)} (4\beta_k)^{-1}} \sum_{k=1}^{N(t)} \frac{1}{4\beta_k} (\mu_k^* - \alpha_k + \lambda_k^*). \tag{5.194}
\]
In this case, \( \lambda_i^*(t) \) and \( \mu_i^*(t) \) are both constants since the state constraint (5.189) is not binding. Thus, the parameter \( \gamma_i(t), i = 1, \ldots, N(t) \) is also constant. We defer the discussion of the price profile at the Stackelberg equilibrium in this case to Case III.

**Case III:** The state constraint (5.189) is binding for some customers given that the sum constraint (5.190) for all customers is also binding.

Denote the set of customers who have state constraint (5.189) binding as \( J \) and other customers belong to the complement \( J^c \). For the customers in \( J \), the input must be zero to meet the state constraint, which is imposed by the solution to customer’s
problem. We have
\[
\dot{x}_j^{(g)}(t) = u_i^*(t) = 0, j \in J_g.
\]
(5.195)

since the constraint (5.190) is also binding, for the customers \(i \in J_g^c\), the inputs must satisfy \(\sum_{i \in J_g} u_i^*(t) = 0\). The optimal inputs can be obtained by summing (5.191) over \(i \in J_g^c\).

\[
\dot{x}_j^{(g)}(t) = u_i^*(t) = \frac{1}{4\beta_i} (\alpha_i - \lambda_i^*(t) - \mu_i^*) + \frac{(4\beta_i)^{-1}}{\sum_{k \in J_g^c} (4\beta_k)^{-1}} \sum_{k \in J_g^c} \frac{1}{4\beta_k} (\mu_k^* - \alpha_k + \lambda_k^*(t)) .
\]
(5.196)

for \(i \in J_g^c\). In this case, the multiplier \(\xi^*(t)\) can be obtained as
\[
\xi^*(t) = \frac{1}{\sum_{i \in J_g^c} (4\beta_k)^{-1}} \sum_{i \in J_g^c} \frac{1}{4\beta_k} (r(t) + \mu_k^* + \lambda_k^* - \alpha_k) .
\]
(5.197)

Since the state constraint (5.189) for customers \(k \in J_g^c\) is not binding, both \(\mu_k^*\) and \(\lambda_k^*\) are constants. It also implies that \(\gamma_k^* = \mu_k^* + \lambda_k^*\) is also constant. Next we consider the price profile for customers in the set \(J_g^c\). Using (5.196), (5.43) and \(\dot{x}_j^{(g)}(t) = \dot{x}_i^{*,D}(t)\), we obtain the price profile as follows.

\[
p_j^{(g)}(t) = \frac{1}{2} (\alpha_j - \lambda_j^*(t) + \mu_j^*) - \frac{1/2}{\sum_{k \in J_g^c} (4\beta_k)^{-1}} \sum_{k \in J_g^c} \frac{1}{4\beta_k} (\mu_k^* + \lambda_k^* - \alpha_k) .
\]
(5.198)

for \(j \in J_g^c\).

Note that if \(J_g^c = \emptyset\), then the price profile is the Stackelberg equilibrium price for the case II.

For the customers \(j \in J_g\), the Lagrange multipliers \(\mu_j^*(t), j \in J_g\) can be obtained
by (5.191) with $u^*_j(t) = 0$ and also (5.197). This is given as

$$
\mu^*_j(t) = \xi^*(t) - (r(t) - \alpha_j + \lambda^*_j(t))
= \frac{1}{\sum_{k \in J_g} (4\beta_k)^{-1} \sum_{k \in J_g} \frac{1}{4\beta_k} (\mu^*_k + \lambda^*_k - \alpha_k) - (\lambda^*_j(t) - \alpha_j)}
$$

(5.199)

for any $j \in J_g$. We can obtain the parameter $\gamma_j(t) = \mu^*_j(t) + \lambda^*_j(t)$,

$$
\gamma_j(t) = \frac{1}{\sum_{k \in J_g} (4\beta_k)^{-1} \sum_{k \in J_g} \frac{1}{4\beta_k} (\gamma_k - \alpha_k) + \alpha_j}.
$$

(5.200)

In this case, the parameter $\gamma_j, j \in J_g$ depends on the parameters of the customer set $J_g^c$. Note that for customers $k \in J_g^c$, $\gamma_k$ is constant since their state constraint is not binding. Thus, $\gamma_j, j \in J_g$ is also constant. The multipliers $\mu^*(t) \geq 0, \lambda^*_j(t) \geq 0$ since the state constraint is binding for $j \in J_g$ in this case. The multiplier $\mu^*_j(t)$ and $\lambda^*_j(t)$ can take any forms that satisfy $\gamma^*_j = \mu^*_j(t) + \lambda^*_j(t)$. One candidate for $\mu^*_j(t)$ and $\lambda^*_j(t)$ are constants values such that $\gamma^*_j = \mu^*_j + \lambda^*_j$.

For customers $j \in J_g$, to achieve Stackelberg equilibrium, the price profile must take appropriate forms such that $\dot{x}^*_i,D(t) = 0$. Using (5.43), the profile of price is given as

$$
p_i^{(g)}(t) = \alpha_i - \lambda_i^*.
$$

(5.201)

**Case IV:** the sum constraint (5.190) is not binding while the state constraint (5.189) is binding for every customer. This case never happens because when (5.189) holds for every customer, every charging rate at Stackelberg equilibrium remains zero, which implies that the sum constraint (5.190) is binding.

Summarizing all the four cases discussed above, we can characterize the charging rate and the price profile at the Stackelberg equilibrium in (5.178).

Next, we move on to solve the price profile $p_i^{(g)}(t)$. With the charging rate profile
\(\dot{x}_i^{(g)}(t)\) determined, it remains to determine \(p_i(t)\) and \(\lambda_i(t)\). We recast the station’s problem as follows.

\[
\max_{p_i(t), i = \{1, 2, \ldots, N(t)\}} \int_0^T \sum_{i=1}^{N(t)} (p_i(t) - r(t)) \dot{x}_i^{(g)}(t) dt.
\]

s.t. \(\int_0^T (\alpha_i - p_i(t)) \dot{x}_i^{(g)}(t) - \beta_i \dot{x}_i^{(g)}(t)^2 dt \geq C_i, i = 1, 2, \ldots, N(t). \) \quad (5.202)

Since \(\dot{x}_i^{(g)}(t), r(t)\) are given, we can model the scalar \(\int_0^T p_i(t) \dot{x}_i^{(g)}(t) dt\) as a decision variable, denoted as \(s_i\). The optimization problem left with us is a linear program with \(N\) variables.

\[
\max_{p_i(t), i = \{1, 2, \ldots, N(t)\}} \sum_{i=1}^{N(t)} s_i - \int_0^T r(t) \dot{x}_i^{(g)}(t) dt.
\]

s.t. \(s_i \leq \int_0^T (\alpha_i \dot{x}_i^{(g)}(t) - \beta_i \dot{x}_i^{(g)}(t)^2(t) - C_i, i = 1, 2, \ldots, N(t). \) \quad (5.203)

At the optimal solution, the equality of each constraint holds. This implies that at Stackelberg equilibrium, the utility of the customer achieves the lower bound.

Consider the utility of any customer \(i\), at Stackelberg equilibrium, of the customer \(i\). Using (5.43) and note that the utility reaches the lower bound \(C_i\), the utility can be written as

\[
\int_0^T \lambda_i^*(\dot{x}_i^{(g)}(t) + \beta_i (\dot{x}_i^{(g)}(t))^2 dt = \lambda_i^*(X - x_0) + \int_0^T (\dot{x}_i^{(g)}(t))^2 dt = C_i. \) \quad (5.204)

Thus, we can characterize \(\lambda_i^*\) as (5.180).

\(\Box\)

**Example 12.** Consider the scenario of three customers who arrive at the station to charge the vehicles at \(t_{1,0} = .1, t_{2,0} = .5, t_{3,0} = 1.6\), respectively. The customers’ parameters are \(\alpha_i = 1, \beta = .1, i = 1, 2, 3\). The initial values of the SoC of both vehicle are \(x_{1,0} = 0.1, x_{2,0} = .5, t_{3,0} = 1.6\). The final SoC levels are \(X_1 = 15, X_2 = 13.5, X_3 = 186\).
10.5. Here, the wholesale electricity price profile is given as \( r(t) = \sin(5t) + 1.1, t \in [0, T] \).

The SoC level of each customer \( x_i(t), i = 1, 2, \ldots, N \) over the time period \([0, T]\) are illustrated in Figure 5.10.

![The state profiles](image)

Figure 5.10. SoC levels at the Stackelberg equilibrium: Batch Arrival

The charging rate profiles of the three vehicles and the sum of the charging rates \( \sum_{i=1}^{3} x_i^{(g)}(t) \) are illustrated in Figure 5.11 and Figure 5.12. We can observe that the sum of the charging rate profiles at the Stackelberg equilibrium remains nonnegative over the entire time horizon.
The price profile at Stackelberg equilibrium is computed and illustrated in Figure 5.13.

![Diagram showing charging rate profiles at Stackelberg equilibrium](image)

Figure 5.11. Charging rate profiles at Stackelberg equilibrium: batch arrival

Next, we analyze the social optimal gap between the Stackelberg equilibrium and competitive equilibrium in this batch arrival case. We need the following technical lemma.

**Lemma 5.5.9.** Consider the batch arrival Stackelberg problem and the competitive equilibrium problem. Suppose all customers have the same parameters $\alpha_i = \alpha, \beta_i = \beta$ for all $i = 1, 2, \ldots, N$. Also suppose that in the solutions of both cases, none of the
customer has the state constraint binding $x_i(t) > L_i x_{i,0}$. Denote the time interval $S = \{ t \in [0,T] | \sum_{i}^{N(t)} \dot{x}(g)(t) = 0 \}$ and $E = \{ t \in [0,T] | \sum_{i}^{N(t)} \dot{x}^*(t) = 0 \}$.

The following inequality holds.

$$E \subseteq S. \quad (5.205)$$

Moreover, the following inequality holds.

$$0 \leq \sum_{i=1}^{N} (\gamma_i^* - \gamma_i^{*,1}) \leq \frac{2\beta}{l_e} \sum_{i=1}^{N} (X_i - x_{i,0}). \quad (5.206)$$

where $0 < l_e \leq T$, $l_e = |E|$.

Proof. Given that no customer has the state constraint binding and the parameters...
are the same for all customers, we can consider the integral of the sum of the charging rates at both equilibria.

\[
\int_0^T \sum_{i=1}^N \dot{x}_i^*(t) \, dt = \int_{\mathcal{E}} \frac{1}{2\beta} \sum_{i=1}^N (\alpha - r(t) - \gamma_{i,0}) \, dt \tag{5.207}
\]

\[
\int_0^T \sum_{i=1}^N \dot{x}_i^{(g)}(t) \, dt = \int_{\mathcal{S}} \frac{1}{4\beta} \sum_{i=1}^N (\alpha - r(t) - \gamma_{i,1}) \, dt \tag{5.208}
\]

By replacing \(r(t), \alpha, \gamma_0^*, \gamma_1^*\) with \(r'(t) = Nr(t), \gamma_1' = \sum_{i=1}^N \gamma_{i,1}, \gamma_0' = \sum_{i=1}^N \gamma_{i,0}\), we can use the same argument in the proof of Lemma 5.5.3 to prove this result. The procedures are omitted.

Next, we consider the social optimality gap between the solutions at the Stackelberg equilibrium and the competitive equilibrium in the batch arrival case. The gap

Figure 5.13. Price profiles at Stackelberg equilibrium: batch arrival
is obtained under the following assumptions on the homogeneity of the customers.

**Assumption 13.** Consider the batch arrival Stackelberg problem and the competitive equilibrium problem.

- All customers has the same parameters $\alpha_i = \alpha, \beta_i = \beta$ for all $i = 1, 2, \ldots, N$.
- In the solutions of both cases, none of the customer has the state constraint (5.175) binding .
- All the customers arrive at the station at the same time $t = 0$.
- The Lagrange multipliers in the competitive equilibrium are equal $\lambda_{i,0} = \lambda_0, i = 1, 2, \ldots, N$ and Lagrange multipliers in the Stackelberg equilibrium are equal $\lambda_{i,1} = \lambda_1, i = 1, 2, \ldots, N$.

By Assumption 13, we state the following result on the increase of the utility obtained by the station in the Stackelberg setting.

**Proposition 5.5.10.** Consider the batch arrival Stackelberg problem and the competitive equilibrium problem under the Assumption 13. Denote the utility of the station at the Stackelberg equilibrium as $K_s^{(g)}$ and the utility of the station at the competitive equilibrium as $K_s^*$. If the price profiles are a set of trajectories such that $\mu_{i,1} = \mu_{i,0}, i = 1, 2, \ldots, N$, then the following inequality holds,

$$K_s^{(g)} - K_s^* > 0. \quad (5.209)$$

**Proof.** Consider the station’s utility $K_s^{(g)}$, which can be written as

$$K_s^{(g)} = \int_0^T \sum_{i=1}^N (\alpha_i - p_i(t)) \dot{x}_i^{(g)}(t) - \beta (\dot{x}_i^{(g)}(t))^2 dt. \quad (5.210)$$

By Assumption 13, we know that the state constraint is not binding for every customer, then we know that the multipliers $\lambda_{i,0}, \lambda_{i,1}, \mu_{i,0}, \mu_{i,1}$ are all constants. Using the property that $p_i(t) = \frac{1}{2}(\alpha_i - \lambda_{i,1}^*(t) + \mu_{i,1}^*(t) + r(t))$ for $t \in S$, and using the
property that \( p_i^*(t) = \alpha - \lambda_{i,0}^* - 2\beta \dot{x}^*(g)(t) \) for each \( i \), we obtain

\[
K_s^{(g)} = \sum_{i=1}^{N} \int_{S_c} (p_i(t) - r(t)) \dot{x}_i^{(g)}(t) \, dt + \sum_{i=1}^{N} \int_{S_c} (p_i(t) - r(t)) \dot{x}^{(g)}(t) \, dt
\]

\[
= \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt + \sum_{i=1}^{N} \mu_{i,1}(X_i - x_{i,0})
\]

\[
+ \sum_{i=1}^{N} \int_{S_c} (\alpha - \lambda_{i,0} - 2\beta \dot{x}^{(g)}(t) - r(t)) \dot{x}^{(g)}(t) \, dt. \tag{5.211}
\]

Use the assumption that \( \lambda_{1,i} = \lambda_1, i = 1, 2, \ldots, N \), then we obtain

\[
K_s^{(g)} = \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt + \sum_{i=1}^{N} \mu_{i,1}(X_i - x_{i,0}) +
\]

\[
\int_{S_c} (\alpha - \lambda_1 - r(t)) \sum_{i=1}^{N} \dot{x}_i^{(g)}(t) - 2\beta \sum_{i=1}^{N} (\dot{x}_i^{(g)}(t))^2 \, dt
\]

\[
= \sum_{i=1}^{N} \mu_{i,1}(X_i - x_{i,0}) + \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt - \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt \tag{5.212}
\]

Since \( \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt - \sum_{i=1}^{N} \int_{S_c} 2\beta (\dot{x}_i^{(g)}(t))^2 \, dt > 0 \), we have

\[
K_s^{(g)} > \sum_{i=1}^{N} \mu_{i,1}(X_i - x_{i,0}) \tag{5.213}
\]

Consider the utility of the station at the competitive equilibrium. Note that in the solution of the price profile at the competitive equilibrium, we have \( \lambda_{i,0} = \lambda_0, i = 1, \ldots, N \). Using the property that \( p_i^*(t) = \alpha - \lambda_{i,0}^* - 2\beta \dot{x}^*(t) \) for each \( i \), we can rewrite
the station’s utility at the competitive equilibrium as

\[
K^s = \int \sum_{i=1}^{N} \mu_{i,0} \hat{x}^*_{i}(t) dt + \int E_c \sum_{i=1}^{N} (\alpha - \lambda^*_{i,0} - 2\beta \hat{x}^{(g)}(t) - r(t)) \hat{x}^*_{i}(t) dt
\]

\[
= \int \sum_{i=1}^{N} \mu_{i,0} \hat{x}^*_{i}(t) dt + \int (\alpha - \lambda^*_{0} - r(t)) \sum_{i=1}^{N} \hat{x}^*_{i}(t) - \sum_{i=1}^{N} \int 2\beta (\hat{x}^{(g)}(t))^2 dt
\]

\[
= \sum_{i=1}^{N} \mu_{i,0} (X_{i,0} - x_{i,0}) - 2\beta \int E_c \sum_{i=1}^{N} (\hat{x}^{(g)}(t))^2 dt
\]

(5.214)

Since \( \int E_c \sum_{i=1}^{N} (\hat{x}^{(g)}(t))^2 dt > 0 \), the station’s utility at the competitive equilibrium can be upper bounded as

\[
K^s < \sum_{i=1}^{N} \mu_{i,0} (X_{i,0} - x_{i,0}).
\]

(5.215)

If \( \mu_{i,0} = \mu_{i,1}, i = 1, 2, \ldots, N \), then

\[
K^{(g)} > K^s.
\]

(5.216)

Thus, we show that the utility of the station is increased in the Stackelberg equilibrium in the batch arrival scenario.

Proposition 5.5.11. Given that the Assumption 13 holds. Denote the customer’s utilities obtained at the competitive equilibrium as \( C_{i,0}, i = 1, 2, \ldots, N \) and the customer’s utilities obtained at the Stackelberg equilibrium as \( C_{i,1}, i = 1, 2, \ldots, N \). Suppose the price profiles are a set of trajectories such that \( \sum_{i=1}^{N} C_{i,0} = \sum_{i=1}^{N} C_{i,1} \). Define the gap between the social welfare at the competitive equilibrium \( W_c \) and the social welfare at the Stackelberg equilibrium \( W_g \) as \( \Delta W = W_c - W_g \). This social optimality
gap satisfies the following inequality.

\[
\Delta W \leq \frac{2\beta}{l_e} \sum_{i} (X_i - x_0)(X_m - x_{m,0}) + 3 \sum_{i=1}^{N} \beta(X_i - x_{i,0})^2
\]

\[
+ 2 \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0}) + \frac{1}{4\beta}(\gamma_{i,0} - \gamma_{i,1})^2l_e
\]

(5.217)

where \(0 < l_e \leq T\), \(l_e = |\mathcal{E}|\) and \(\mathcal{E}\) is time domain such that the charging rate at the competitive equilibrium \(\sum_{i=1}^{N} \dot{x}_i(t) > 0\) over the time period \([0, T]\). The index \(m\) is selected from \(1, 2, \ldots, N\) such that

\[
\sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_m - x_{m,0}) \geq \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0})
\]

Proof. The social welfare of the Stackelberg equilibrium and competitive equilibrium are represented as

\[
W_c = K_s^* + \sum_{i=1}^{N} C_{i,0}, \quad W_g = K_s^{(g)} + \sum_{i=1}^{N} C_{i,1}.
\]

(5.218)

where \(K_s^*\), \(K_s^{(g)}\) are the utilities of the station obtained at the competitive equilibrium and Stackelberg equilibrium. The social optimality gap is

\[
\Delta S = K_s^* - K_s^{(g)}.
\]

(5.219)

By (5.215), the station’s utility \(K_s^*\) is upper bounded by

\[
K_s^* < \sum_{i=1}^{N} \mu_{i,0}(X_i - x_{i,0}).
\]

(5.220)

By (5.213), the station’s utility \(K_s^{(g)}\) is lower bounded by

\[
K_s^{(g)} > \sum_{i=1}^{N} \mu_{i,1}(X_i - x_{i,1}).
\]

(5.221)
By \( \gamma_{i} = \lambda_{i} + \mu_{i}, \gamma_{0,i} = \lambda_{0,i} + \mu_{0,i} \) and combine the representations of \( K^* \) and \( K^{(g)} \), the social optimality gap can be written as

\[
\Delta W = K^* - K^{(g)} < \sum_{i=1}^{N} (\mu_{i,0} - \mu_{i,1})(X_i - x_{i,0})
\]

(5.222)

\[
= \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0}) - \sum_{i=1}^{N} (\lambda_{i,0} - \lambda_{i,1})(X_i - x_{i,0})
\]

(5.223)

The first term of the right hand side of \( \Delta S \) can be upper bounded as follows. Note that we can select \( m \) from \( 1, \ldots, N \) such that \( \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_m - x_{m,0}) \geq \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0}) \). Also use (5.206), we obtain

\[
\sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0}) \leq \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_m - x_{m,0})
\]

(5.224)

Next, we show the upper bound of the term \( -\sum_{i=1}^{N} (\lambda_{i,0} - \lambda_{i,1})(X_i - x_{i,0}) \) in \( \Delta S \). Denote the sum of the customer’s utility at the competitive equilibrium as \( K^*_c \), which can be obtained as

\[
K^*_c = \beta \sum_{i=1}^{N} \int_{0}^{T} (\dot{x}^*(t))^2 dt + \sum_{i=1}^{N} (\lambda_{i,0}(X_i - x_{i,0})).
\]

(5.225)

Denote the sum of the customer’s utility at the competitive equilibrium as \( K^{(g)}_c \), which can be obtained as

\[
K^{(g)}_c = 2\beta \sum_{i=1}^{N} \int_{0}^{T} (\dot{x}^{(g)}(t))^2 dt + \sum_{i=1}^{N} (\lambda_{i,1}(X_i - x_{i,0})).
\]

(5.226)
By $K^*_c = K_c^{(g)}$, we obtain

$$
\sum_{i=1}^{N}(\lambda_{i,0} - \lambda_{i,1})(X_i - x_{i,0}) = 2\beta \sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt - \beta \sum_{i=1}^{N} \int_0^T (\dot{x}^*(t))^2 dt. \tag{5.227}
$$

By $|S| > |E|$ in Lemma 5.5.9 we have

$$
\sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt \geq \sum_{i=1}^{N} \int_{S}(\dot{x}^{(g)}(t))^2 dt \geq \sum_{i=1}^{N} \int_{E}(\dot{x}^{(g)}(t))^2 dt \tag{5.228}
$$

By (5.154), we obtain

$$
\sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt \geq \frac{1}{4} \sum_{i=1}^{N} \int_{E}(\dot{x}^{*}(t))^2 dt + \frac{1}{2\beta} \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0})
\quad + \frac{1}{16\beta^2}(\gamma_{i,0} - \gamma_{i,1})^2 |E|. \tag{5.229}
$$

With (5.229) and (5.227), we have

$$
\sum_{i=1}^{N}(\lambda_{i,0} - \lambda_{i,1})(X_i - x_{i,0}) = \tag{5.230}
$$

$$
4\beta \sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt - \sum_{i=1}^{N} \int_0^T (\dot{x}^{*}(t))^2 dt - 2\beta \sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt
\quad \geq \sum_{i=1}^{N} \int_{E}\beta(\dot{x}^{*}(t))^2 dt - \sum_{i=1}^{N} \int_0^T \beta(\dot{x}^{*}(t))^2 dt + 2 \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0})
\quad + \frac{1}{4\beta}(\gamma_{i,0} - \gamma_{i,1})^2 l_e - 2\beta \sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt
\quad \geq - \sum_{i=1}^{N} \int_0^T \beta(\dot{x}^{*}(t))^2 dt - 2 \sum_{i=1}^{N} (\gamma_{i,0} - \gamma_{i,1})(X_i - x_{i,0})
\quad - \frac{1}{4\beta}(\gamma_{i,0} - \gamma_{i,1})^2 l_e - 2\beta \sum_{i=1}^{N} \int_0^T (\dot{x}^{(g)}(t))^2 dt
$$

where the last inequality holds because $\sum_{i=1}^{N} \int_{E} \beta(\dot{x}^{*}(t))^2 dt > 0$. Using Cauchy-
Schwartz inequality, we can obtain the following inequality from (5.230),

\[ \sum_{i=1}^{N} (\lambda_i - \lambda_i, 1)(X_i - x_{i,0}) \geq - \sum_{i=1}^{N} \beta \left( \int_{0}^{T} \dot{x}^*(t) dt \right)^2 - 2 \sum_{i=1}^{N} (\gamma_i, 0 - \gamma_i, 1)(X_i - x_{i,0}) \]

\[ - \frac{1}{4\beta} (\gamma_i, 0 - \gamma_i, 1)^2 l_e - 2\beta \sum_{i=1}^{N} \left( \int_{0}^{T} \dot{x}^g(t) dt \right)^2 \]

\[ = -3 \sum_{i=1}^{N} \beta (X_i - x_{i,0})^2 - 2 \sum_{i=1}^{N} (\gamma_i, 0 - \gamma_i, 1)(X_i - x_{i,0}) - \frac{1}{4\beta} (\gamma_i, 0 - \gamma_i, 1)^2 l_e \quad (5.231) \]

By the inequalities (5.231) and (5.224), the upper bound of the social optimality gap is obtained as follows.

\[ \Delta W \leq \frac{2\beta}{l_e} \sum_{i}^{N} (X_i - x_0)(X_m - x_{m,0}) + 3 \sum_{i=1}^{N} \beta (X_i - x_{i,0})^2 + 2 \sum_{i=1}^{N} (\gamma_i, 0 - \gamma_i, 1)(X_i - x_{i,0}) + \frac{1}{4\beta} (\gamma_i, 0 - \gamma_i, 1)^2 l_e \quad (5.232) \]

5.6 Conclusions

In this chapter, we studied the dynamic equilibria in the commercial charging station business. The station and the customers are viewed as market players to reach an agreement on the trajectory of the charging over a given period. Under the assumption that the price is exogenous, the competitive equilibrium is characterized to maximize each of the utilities of both parties simultaneously. We considered the one station and one customer case and also extended the results to the case with multiple customers who arrives in different time during a given time interval. Optimal control strategies are adopted to solve the dynamic equilibrium problem, which is seldom addressed in the market analysis of the power systems. The competitive equilibrium solution is also shown to be efficient, which maximizes the social welfare. Moreover,
the scenario that the station has enough market power to affect the price of charging is also considered. A Stackelberg game framework is used to study such a scenario. The station is modeled as a leader and the customers are modeled as followers. The charging rate profile at the Stackelberg equilibrium is characterized, with a guarantee that each customer receive a nonnegative utility. Since the Stackelberg is not socially optimal, we derived the upper bound of the socially optimal gap as compared to the competitive equilibrium. Future work includes the consideration of uncertainty in the arrival pattern of the customers as well as uncertainty of the wholesale power price.
In this dissertation, we discussed two problems in the distributed systems. In the first set of problems, we focused on design of distributed systems with multiple components (sensor, actuator, controller) that interact across the communication networks that limit information exchange. In the second set of problems, we considered the design of markets and economic incentives for various agents to align their goals with the planner’s goal, in the specific content of commercial electric vehicle charging stations.

In the first area, we obtained results on stabilizability of decentralized systems in the presence of the data erasure channels in Chapter 2. The system is modeled as an information transmission network that consists of both signaling and erasure channels. Necessary and sufficient conditions for stabilizability of the system are obtained. Further, we obtained the stabilizability condition on control systems with feedback signals sent over a Gaussian MAC channel in Chapter 3. Necessary conditions for stabilizability that connect the system eigenvalues and the Gaussian MAC channel are obtained. Moreover, we propose a periodic control scheme to stabilize the system that achieves new points in the rate region sufficient for stabilizability. An interesting observation is that the sufficient rate region with correlated process noise present in the system is larger than the uncorrelated process noise, since the correlation of the process noise helps information transmission over the Gaussian MAC channel.

The work discussed in this area opens up new future directions for further investigation of the effect of communication network in the decentralized systems. In the
first result on the erasure channels in decentralized systems, an actuator coordinating mechanism is needed to avoid conflicts between multiple actuators that may attempt to change the value of the same mode. If a proper scheduling or control scheme can be designed to avoid using this mechanism, it will be a great improvement to the distributed system design. Since such conflicts of the actuators are seldom studied in the literature of decentralized systems, new perspectives or approach are highly required to achieve this goal.

Another direction is to study the signaling process established through plants with unbounded process noises and the impact of its channel capacity on stabilizing the distributed systems. This direction has not been well studied. Most existing literature only studied the signaling through plants without any noise or plants with bounded noises. To build a signaling process through noisy plants, new encoding and decoding scheme are needed, as well as the characterization of the channel capacity of such implicit channels. In a large scale distributed system, it will be interesting if one plant is signaling to more than one plants, or two plants are signaling to one plants. These scenarios are similar to transmit information over Gaussian multi-user channels (broadcast, multiple access). Results from networked control over Gaussian networks and distributed control systems can used jointly to solve such problems. The condition for stabilizability that connects the eigenvalues of the subsystems and the channel capacity of the signaling channels will be exciting to both the distributed control and communication communities.

The second problem we consider is design of markets and economic incentives for various agents to align their goals with the social planner’s goal, which mainly focuses on the analysis of the power system and electric vehicle charging infrastructures. We obtained results on the Nash equilibrium of the pricing games of the charging stations in different locations in Chapter 4. Sufficient conditions on the existence of the Nash equilibrium of the pricing games are obtained. This work considers various layouts
of the possible locations of the game, including the congestion and queueing effect of
the stations and its pricing strategies.

We further consider the charging station business from the perspective of the real-
time scale in Chapter 5. A competitive equilibrium in the dynamic setting is studied
over the time horizon of charging a vehicle, with the power retail price and charging
price varying in real-time. The competitive equilibrium in the form of trajectories is
characterized under the assumption that the charging price is exogenous. The study is
also extended to consider the case that the station has enough market power to affect
the price, which leads to a Stackelberg equilibrium scenario. The main contribution
in this aspect is to apply the classic optimal control theory such as Pontryagin’s
minimum principle to characterize the trajectories which are the optimal solution to
more than one optimal problems. The results are first obtained in the simple case
that involves one station and one PEV customer and then extended to the case with
multiple customers.

This work can be extended along several directions. It will be interesting to
extend the static locations and pricing solution to the case with multiple charging
stations or multiple customer nodes. Since we solve two nonlinear programs jointly
to get the candidate of the Nash equilibrium, a new approach might be needed for
the general case to escape solving many nonlinear programs for multiple station case.
Moreover, it would be interesting to consider the case when prices are not static. In
our formulation, the game of static prices is analyzed based on the long-term average
analysis of the market, which assumes that the customer allocation satisfies the Nash
equilibrium after enough time. However, in reality, the queues of stations will keep
changing and the customers are always willing to join the shorter queue. A dynamic
price scheme based on the previous price and current number of the customers waiting
in the queue will be more desirable. Analysis tools such as semi-Markov games [64]
will be needed to design such a dynamic price scheme.
The results of the real-time pricing profile obtained in Chapter 5 can be extended to consider the stochastic behavior of the electric vehicle arrivals. In our result for the competitive equilibrium case, the station has access to all the information of the arrivals of the electric vehicles. In reality, uncertainties in the vehicle arrival times will impact on the scheduling and charging profiles planning. Thus, a fruitful direction of future work is to find a stochastic analysis approach to find a charging trajectory or pricing scheme to achieve an equilibrium in this case. The stations can be assumed to have various information about the arrival pattern of the customers. Other uncertainties, such as the power retail prices are stochastic can also be considered in the similar framework.


